Section 10.6

1.) $$\lim_{n\to\infty} \frac{\frac{(n+1)^2}{3^{n+1}}}{\frac{n^2}{3^n}} = \lim_{n\to\infty} \frac{(n+1)^2}{n^2} \cdot \frac{3^n}{3^{n+1}}$$

$$= \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^2 \cdot \frac{1}{3} = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^2 \cdot \frac{1}{3} = (1)^2 \cdot \frac{1}{3} = \frac{1}{3} < 1,$$

so $\sum_{n=1}^{\infty} \frac{n^2}{3^n}$ converges by the ratio test.

2.) $$\lim_{n\to\infty} \frac{\frac{(n+2)^2}{(n+1)2^{n+1}}}{\frac{(n+1)^2}{n2^n}} = \lim_{n\to\infty} \frac{(n+2)^2 n}{(n+1)^3} \cdot \frac{2^n}{2^{n+1}}$$

$$= \lim_{n\to\infty} \left(\frac{n+2}{n+1}\right)^2 \cdot \frac{n}{n+1} \cdot \frac{1}{2} = \lim_{n\to\infty} \left(\frac{1+\frac{2}{n}}{1+\frac{1}{n}}\right)^2 \left(\frac{1}{1+\frac{1}{n}}\right) \cdot \frac{1}{2}$$

$$= \left(\frac{1+0}{1+0}\right)^2 \left(\frac{1}{1+0}\right) \cdot \frac{1}{2} = \frac{1}{2} < 1,$$ so $\sum_{n=1}^{\infty} \frac{(n+1)^2}{n2^n}$

converges by the ratio test.

3.) $$\lim_{n\to\infty} \frac{\frac{(n+1)\ln(n+1)}{3^{n+1}}}{\frac{n\ln n}{3^n}} = \lim_{n\to\infty} \left(\frac{n+1}{n}\right) \frac{\ln(n+1)}{\ln n} \cdot \frac{3^n}{3^{n+1}}$$

$\frac{\infty}{\infty}$

$$= \lim_{n\to\infty} \left(1+\frac{1}{n}\right) \frac{\ln(n+1)}{\ln n} \cdot \frac{1}{3} = \lim_{n\to\infty} \left(1+\frac{1}{n}\right) \cdot \frac{\frac{1}{n+1}}{\frac{1}{n}} \cdot \frac{1}{3}$$

$$= \lim_{n\to\infty} \left(1+\frac{1}{n}\right) \cdot \frac{n}{n+1} \cdot \frac{1}{3} = \lim_{n\to\infty} \left(1+\frac{1}{n}\right) \frac{1}{1+1/n} \cdot \frac{1}{3}$$

$$= 1 \cdot 1 \cdot \frac{1}{3} = \frac{1}{3} < 1,$$ so $\sum_{n=1}^{\infty} \frac{n\ln n}{3^n}$ converges

by the ratio test.

4.) $\lim_{n\to\infty} \dfrac{\frac{(n+1)!}{3^{n+1}}}{\frac{n!}{3^n}} = \lim_{n\to\infty} \dfrac{(n+1)!}{n!} \cdot \dfrac{3^n}{3^{n+1}}$

$= \lim_{n\to\infty} \dfrac{n+1}{3} = \infty > 1$, so $\sum_{n=1}^{\infty} \dfrac{n!}{3^n}$

diverges by the ratio test.

5.) $\lim_{n\to\infty} \dfrac{\frac{(2(n+1)+1)(2^{n+1}+1)}{3^{n+1}+1}}{\frac{(2n+1)(2^n+1)}{3^n+1}} = \lim_{n\to\infty} \dfrac{2n+3}{2n+1} \cdot \dfrac{2^{n+1}+1}{2^n+1} \cdot \dfrac{3^n+1}{3^{n+1}+1}$

$= \lim_{n\to\infty} \dfrac{1+3/2n}{1+1/2n} \cdot \dfrac{2+1/2^n}{1+1/2^n} \cdot \dfrac{1+1/3^n}{3+1/3^n} = (1)(2)(1/3) = \frac{2}{3} < 1$,

so $\sum_{n=1}^{\infty} \dfrac{(2n+1)(2^n+1)}{3^n+1}$ converges by the ratio test.

6.) $\lim_{n\to\infty} \dfrac{\frac{(n+1)!}{(n+1)^{n+1}}}{\frac{n!}{n^n}} = \lim_{n\to\infty} \dfrac{(n+1)!}{(n+1)^{n+1}} \cdot \dfrac{n^n}{n!}$

$= \lim_{n\to\infty} \dfrac{(n+1)}{(n+1)^{n+1}} \cdot \dfrac{n^n}{1} = \lim_{n\to\infty} \dfrac{n^n}{(n+1)^n} = \lim_{n\to\infty} \dfrac{1}{\frac{(n+1)^n}{n^n}}$

$= \lim_{n\to\infty} \dfrac{1}{(1+\frac{1}{n})^n} = \dfrac{1}{e} < 1$, so $\sum_{n=1}^{\infty} \dfrac{n!}{n^n}$ converges by the ratio test

7.) $\sum_{n=1}^{\infty} \dfrac{2^n}{(n+1)^n}$

a.) $\lim_{n\to\infty} \dfrac{2^{n+1}}{(n+2)^{n+1}} \cdot \dfrac{(n+1)^n}{2^n} = \lim_{n\to\infty} 2 \cdot \dfrac{(n+1)^n}{(n+2)^n} \cdot \dfrac{1}{n+2}$

$= \lim_{n\to\infty} \dfrac{2}{n+2} \cdot \dfrac{1}{\left(\frac{n+2}{n+1}\right)^n} = \lim_{n\to\infty} \dfrac{2}{n+2} \cdot \dfrac{1}{\left[\left(1+\frac{1}{n+1}\right)^{(n+1)}\right]^{\frac{n}{n+1}}}$

$= 0 \cdot \frac{1}{e^1} = 0 < 1$, so $\sum_{n=1}^{\infty} \frac{2^n}{(n+1)^n}$ converges by the ratio test.

b.) $\lim_{n\to\infty} \left(\frac{2^n}{(n+1)^n}\right)^{1/n} = \lim_{n\to\infty} \frac{2}{n+1} = 0 < 1$, so $\sum_{n=1}^{\infty} \frac{2^n}{(n+1)^n}$ converges by the root test.

c.) $\left\{\frac{2^n}{(n+1)^n}\right\}$: $\frac{2^1}{2^1}, \frac{2^2}{3^2}, \frac{2^3}{4^3}, \frac{2^4}{5^4}, \cdots$ so that $\frac{2^n}{(n+1)^n} \le \frac{2^n}{3^n}$ for $n \ge 2$; since $\sum_{n=2}^{\infty} \left(\frac{2}{3}\right)^n$ is a convergent geometric series ($r = \frac{2}{3}$, $-1 < r < 1$), then $\sum_{n=1}^{\infty} \frac{2^n}{(n+1)^n}$ converges by the comparison test.

8.) $\sum_{n=1}^{\infty} \frac{2^n}{n^5}$ a.) $\lim_{n\to\infty} \frac{2^{n+1}}{(n+1)^5} \cdot \frac{n^5}{2^n} = \lim_{n\to\infty} 2 \cdot \left(\frac{n}{n+1}\right)^5$

$= 2 \cdot (1)^5 = 2 > 1$, so $\sum_{n=1}^{\infty} \frac{2^n}{n^5}$ diverges by the ratio test.

b.) $\lim_{n\to\infty} \frac{2^n}{n^5} \overset{\text{"}\frac{\infty}{\infty}\text{"}}{=} \lim_{n\to\infty} \frac{2^n \cdot \ln 2}{5n^4} \overset{\text{"}\frac{\infty}{\infty}\text{"}}{=} \lim_{n\to\infty} \frac{2^n (\ln 2)^2}{20n^3}$

$\overset{\text{"}\frac{\infty}{\infty}\text{"}}{=} \lim_{n\to\infty} \frac{2^n (\ln 2)^3}{60n^2} \overset{\text{"}\frac{\infty}{\infty}\text{"}}{=} \lim_{n\to\infty} \frac{2^n (\ln 2)^4}{120n}$

$\overset{\text{"}\frac{\infty}{\infty}\text{"}}{=} \lim_{n\to\infty} \frac{2^n (\ln 2)^5}{120} = \infty \ne 0$, so $\sum_{n=1}^{\infty} \frac{2^n}{n^5}$ diverges by the nth term test.

13.) $\lim_{n\to\infty} \left(\frac{n^n}{3^{n^2}}\right)^{1/n} = \lim_{n\to\infty} \frac{n}{3^n} \overset{\text{"}\frac{\infty}{\infty}\text{"}}{=} \lim_{n\to\infty} \frac{1}{3^n \cdot \ln 3} = 0 < 1$

so $\sum_{n=1}^{\infty} \frac{n^n}{3^{n^2}}$ converges by the root test.

14.) $\lim_{n\to\infty}\left[\dfrac{(1+1/n)^n(2n+1)^n}{(3n+1)^n}\right]^{1/n} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)\cdot\left(\frac{2n+1}{3n+1}\right)$

$= 1\cdot\frac{2}{3} = \frac{2}{3} < 1$, so $\displaystyle\sum_{n=1}^{\infty}\frac{(1+1/n)^n(2n+1)^n}{(3n+1)^n}$

converges by the root test.

16.) $\displaystyle\sum_{n=1}^{\infty}\frac{\ln n}{n^2} \le \sum_{n=1}^{\infty}\frac{\sqrt{n}}{n^2} = \sum_{n=1}^{\infty}\frac{1}{n^{3/2}} < \frac{1}{1^{3/2}}+\int_1^{\infty}\frac{1}{x^{3/2}}\,dx$

$= 1+\left.\dfrac{-2}{x^{1/2}}\right|_1^{\infty} = 1+\left(0-\frac{-2}{1}\right) = 3$

20.) $\displaystyle\sum_{n=1}^{\infty}\frac{n}{n^3+1} \le \sum_{n=1}^{\infty}\frac{n}{n^3} = \sum_{n=1}^{\infty}\frac{1}{n^2} < \frac{1}{1^2}+\int_1^{\infty}\frac{1}{x^2}\,dx$

$= 1+\left.\dfrac{-1}{x}\right|_1^{\infty} = 1+\left(0-\frac{-1}{1}\right) = 2$

22.) $\displaystyle\frac{1}{1^2+1}+\frac{2}{2^2+1}+\frac{3}{3^2+1}+\cdots+\frac{m}{m^2+1} > \int_1^{m+1}\frac{x}{x^2+1}\,dx$

$= \frac{1}{2}\ln(x^2+1)\Big|_1^{m+1} = \frac{1}{2}\ln((m+1)^2+1)-\frac{1}{2}\ln 2 \ge 1000 \Rightarrow$

$\ln((m+1)^2+1) \ge \ln 2+2000 \Rightarrow (m+1)^2+1 \ge e^{(\ln 2+2000)} \Rightarrow$

$(m+1)^2 \ge 2e^{2000}-1 \Rightarrow m \ge -1+(2e^{2000}-1)^{1/2} \Rightarrow$

$m \ge 2.7861\times 10^{434}$.

23.) $\displaystyle(1.01)^1+(1.01)^2+(1.01)^3+\cdots+(1.01)^m > \int_1^{m+1}(1.01)^x\,dx$

$= \left.\dfrac{(1.01)^x}{\ln 1.01}\right|_1^{m+1} = \dfrac{(1.01)^{m+1}}{\ln 1.01}-\dfrac{1.01}{\ln 1.01} \ge 1000 \Rightarrow$

$\dfrac{(1.01)^{m+1}}{\ln 1.01} \ge 1000+\dfrac{1.01}{\ln 1.01} \Rightarrow (1.01)^{m+1} \ge \ln 1.01\left(1000+\dfrac{1.01}{\ln 1.01}\right)$

$\Rightarrow (1.01)^{m+1} \ge 10.96 \Rightarrow (m+1)\ln 1.01 \ge \ln 10.96$

$$\Rightarrow m \geq \frac{\ln 10.96}{\ln 1.01} - 1 \approx 239.6$$ so choose $m \geq 240$.

26.) a.) $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges by harmonic series test ;

$$\lim_{n\to\infty} \left(\frac{1}{n}\right)^{1/n} = \lim_{n\to\infty} \frac{1}{n^{1/n}} = \lim_{n\to\infty} n^{-1/n}$$

$$= \lim_{n\to\infty} e^{\ln n^{-1/n}} = \lim_{n\to\infty} e^{-\frac{1}{n}\ln n}$$

$$= e^{\lim_{n\to\infty} \frac{-\ln n}{n}} \overset{\text{"}\frac{\infty}{\infty}\text{"}}{=} e^{\lim_{n\to\infty} \frac{-\frac{1}{n}}{1}} = e^0 = 1 .$$

b.) $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges by the p-series test $(p = 2 > 1)$;

$$\lim_{n\to\infty} \left(\frac{1}{n^2}\right)^{1/n} = \lim_{n\to\infty} \frac{1}{n^{2/n}} = \lim_{n\to\infty} n^{-2/n}$$

$$= \lim_{n\to\infty} e^{\ln n^{-2/n}} = \lim_{n\to\infty} e^{\frac{-2\ln n}{n}}$$

$$= e^{\lim_{n\to\infty} \frac{-2\ln n}{n}} \overset{\text{"}\frac{\infty}{\infty}\text{"}}{=} e^{\lim_{n\to\infty} \frac{-2}{n}} = e^0 = 1 .$$