Section 10.7

1.) $\lim_{n\to\infty} \frac{n}{n+1} = 1$ so that $\lim_{n\to\infty} (-1)^{n+1} \cdot \frac{n}{n+1}$ DNE, i.e., $\lim_{n\to\infty} (-1)^{n+1} \cdot \frac{n}{n+1} \neq 0$, so that the series diverges by the nth term test.

2.) $\lim_{n\to\infty} \frac{1}{1+2^{-n}} = \frac{1}{1+0} = 1$ so that $\lim_{n\to\infty} (-1)^n \cdot \frac{1}{1+2^{-n}}$ DNE, i.e., $\lim_{n\to\infty} (-1)^n \cdot \frac{1}{1+2^{-n}} \neq 0$, so that the series diverges by the nth term test.

3.) $a_n = \frac{1}{\sqrt{n}}$ is $+, \downarrow$, and $\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$, so that $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{\sqrt{n}}$ converges by the alternating series test.

4.) $a_n = \frac{5^n}{n!}$ is $+, \downarrow$ (for $n \geq 6$), and $\lim_{n\to\infty} \frac{5^n}{n!} = 0$, so that $\sum_{n=6}^{\infty} (-1)^{n+1} \cdot \frac{5^n}{n!}$ and $\sum_{n=1}^{\infty} (-1)^{n+1} \cdot \frac{5^n}{n!}$ converge by the alternating series test.

5.) $\frac{3}{\sqrt{1}} - \frac{2}{\sqrt{1}} + \frac{3}{\sqrt{2}} - \frac{2}{\sqrt{2}} + \frac{3}{\sqrt{3}} - \frac{2}{\sqrt{3}} + \cdots = \sum_{n=1}^{\infty} \left(\frac{3}{\sqrt{n}} - \frac{2}{\sqrt{n}} \right)$
$= \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$, which diverges by the p-series test ($p = \frac{1}{2} \leq 1$).

6.) $\lim_{n\to\infty} \sqrt{n} = \infty$ so that $\lim_{n\to\infty} (-1)^{n+1} \sqrt{n}$ DNE, i.e., $\lim_{n\to\infty} (-1)^{n+1} \sqrt{n} \neq 0$, so that the series diverges by the nth term test.

7.) $\lim_{n\to\infty} \frac{n}{2n+1} = \frac{1}{2}$ so that $\lim_{n\to\infty} (-1)^{n+1} \cdot \frac{n}{2n+1}$ DNE, i.e., $\lim_{n\to\infty} (-1)^{n+1} \cdot \frac{n}{2n+1} \neq 0$, so that the series diverges by the nth term test.

8.) $a_n = \frac{1}{n^2}$ is $+$, $\downarrow$, and $\lim_{n\to\infty} \frac{1}{n^2} = 0$, so that $\sum_{n=1}^{\infty} (-1)^{n+1} \cdot \frac{1}{n^2}$ converges by the alternating series test.

9.) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$

a.) $S_5 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} \approx 0.78333$,

$S_6 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} \approx 0.61666$

b.) S_5 is an overestimate since

$$R_5 = \underbrace{\frac{-1}{6} + \frac{1}{7}}_{(-)} + \underbrace{\frac{-1}{8} + \frac{1}{9}}_{(-)} + \underbrace{\frac{-1}{10} + \frac{1}{11}}_{(-)} + \cdots < 0;$$

S_6 is an underestimate since

$$R_6 = \underbrace{\frac{1}{7} + \frac{-1}{8}}_{(+)} + \underbrace{\frac{1}{9} + \frac{-1}{10}}_{(+)} + \underbrace{\frac{1}{11} + \frac{-1}{12}}_{(+)} + \cdots > 0.$$

c.) By a.) and b.) it follows that

$$0.61666 < \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} < 0.78333$$

10.) $\sum_{n=1}^{\infty} (-1)^{n+1} \cdot \frac{1}{n2^n}$

a.) $$S_6 = \frac{1}{1\cdot 2} - \frac{1}{2\cdot 2^2} + \frac{1}{3\cdot 2^3} - \frac{1}{4\cdot 2^4} + \frac{1}{5\cdot 2^5} - \frac{1}{6\cdot 2^6}$$

$$= \frac{1}{2} - \frac{1}{8} + \frac{1}{24} - \frac{1}{64} + \frac{1}{160} - \frac{1}{384}$$

$$\approx 0.40469$$

b.) $$R_6 = \underbrace{\frac{1}{7\cdot 2^7} - \frac{1}{8\cdot 2^8}}_{(+)} + \underbrace{\frac{1}{9\cdot 2^9} - \frac{1}{10\cdot 2^{10}}}_{(+)} + \cdots \quad 0$$

so absolute error $|R_6| < a_7$, i.e.,

$$|R_6| < \frac{1}{7\cdot 2^7} \approx 0.00111 \quad ;$$

since $R_6 > 0$, S_6 is an underestimate and therefore

$$0.40469 < \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n2^n} < 0.40469 + 0.00111 \rightarrow$$

$$0.40469 < \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n2^n} < 0.40580$$

28.) $\sum_{n=1}^{\infty} (-1)^{n+1} \cdot \frac{1}{2^n}$

a.) $\frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \frac{1}{16} + \frac{1}{32} - \frac{1}{64} = S_6 \approx 0.328$

b.) $|R_6| < a_7 = \frac{1}{2^7} \approx 0.008$

c.) $R_6 = \sum_{n=7}^{\infty} (-1)^{n+1} \frac{1}{2^n} = \frac{1}{2^7} - \frac{1}{2^8} + \frac{1}{2^9} - \frac{1}{2^{10}} + \cdots$

$= \frac{1}{2^7} \cdot \left[1 + \left(\frac{-1}{2}\right) + \left(\frac{-1}{2}\right)^2 + \left(\frac{-1}{2}\right)^3 + \cdots \right] = \frac{1}{2^7} \cdot \frac{1}{1 - (-1/2)}$

$= \frac{1}{2^7} \cdot \frac{1}{3/2} = \frac{1}{3 \cdot 2^6} \approx 0.00521$

30.) $\frac{1}{6!} \approx 0.0013$ so $S_6 = 1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} \approx 0.367$, and S_6 estimates the value of $\sum_{n=0}^{\infty} (-1)^n / n!$ with absolute error at most $\frac{1}{6!}$, i.e., to two decimal places.

31.) a.) $\lim_{n\to\infty} \frac{2^{n+1}}{(n+1)!} \cdot \frac{n!}{2^n} = \lim_{n\to\infty} \frac{2}{n+1} = 0 < 1$, so $\sum_{n=1}^{\infty} \frac{n!}{2^n}$ converges by the ratio test.

$$\text{b.)}\ \sum_{n=1}^{\infty}\frac{2^n}{n!}=\frac{2}{1}+\frac{2\cdot 2}{2\cdot 1}+\overbrace{\frac{2\cdot 2\cdot 2}{3\cdot 2\cdot 1}}^{2^3}+\overbrace{\frac{2\cdot 2\cdot 2\cdot 2}{4\cdot 3\cdot 2\cdot 1}}^{2^4}+\overbrace{\frac{2\cdot 2\cdot 2\cdot 2\cdot 2}{5\cdot 4\cdot 3\cdot 2\cdot 1}}^{2^5}+\cdots+\frac{2^{m+2}}{(m+2)!}+\frac{2^{m+3}}{(m+3)!}+\cdots$$

$$\le \frac{2}{1}+\frac{2}{1}+\left(\frac{2}{3}\right)(2)+\left(\frac{2}{3}\right)^2(2)+\left(\frac{2}{3}\right)^3(2)+\cdots+\left(\frac{2}{3}\right)^m(2)+\left(\frac{2}{3}\right)^{m+1}(2)+\cdots$$

if $\left(\frac{2}{3}\right)^m(2)+\left(\frac{2}{3}\right)^{m+1}(2)+\left(\frac{2}{3}\right)^{m+2}(2)+\cdots\le 0.005 \Rightarrow$

$(2)\left(\frac{2}{3}\right)^m\cdot\left[1+\left(\frac{2}{3}\right)+\left(\frac{2}{3}\right)^2+\cdots\right]\le 0.005 \Rightarrow$

$(2)\left(\frac{2}{3}\right)^m\cdot\dfrac{1}{1-\frac{2}{3}}\le 0.005 \Rightarrow (6)\left(\frac{2}{3}\right)^m\le 0.005 \Rightarrow$

$\left(\frac{2}{3}\right)^m\le\dfrac{0.005}{6}\Rightarrow \ln\left(\frac{2}{3}\right)^m\le \ln\left(\dfrac{0.005}{6}\right)\Rightarrow$

$m\ln(2/3)\le \ln\left(\dfrac{0.005}{6}\right)\Rightarrow m\ge\dfrac{\ln\left(\frac{0.005}{6}\right)}{\ln(2/3)}\approx 17.5$;

for geom. series let error start with $m=18$ (and partial sum is S_{17}); then for $\sum_{n=1}^{\infty}\frac{2^n}{n!}$ error starts with $\dfrac{2^{18+2}}{(18+2)!}$ term and partial sum is S_{19} ; thus

$S_{19}=\dfrac{2^1}{1!}+\dfrac{2^2}{2!}+\dfrac{2^3}{3!}+\cdots+\dfrac{2^{19}}{19!}\approx 6.389$

will estimate $\sum_{n=1}^{\infty}\frac{2^n}{n!}$ with error at most 0.005, i.e., to two decimal places.

39.) $\sum_{n=1}^{\infty}a_n$ converges, $a_n>0$, so that $\lim_{n\to\infty}a_n=0$.

a.) $\lim_{n\to\infty}\dfrac{\sin(a_n)}{a_n}=1>0$, so $\sum_{n=1}^{\infty}\sin(a_n)$ converges by the limit comparison test.

b.) $\lim_{n\to\infty}\cos(a_n)=\cos(0)=1\neq 0$, so $\sum_{n=1}^{\infty}\cos(a_n)$ diverges by the nth term test.