

Section 11.1

Taylor series

$$(T) \quad f(x) = a_0 + a_1(x-a) + a_2(x-a)^2 + \dots = \sum_{n=0}^{\infty} a_n(x-a)^n$$

$$(S) \quad a_n = \frac{f^{(n)}(a)}{n!} \quad \text{for } n=0, 1, 2, 3, \dots$$

$$(P) \quad P_n(x; a) = a_0 + a_1(x-a) + a_2(x-a)^2 + \dots + a_n(x-a)^n$$

Taylor polynomial \int terminates at the n th power of $x-a$

$$13.) \quad f(x) = \ln(1+x),$$

$$f'(x) = \frac{1}{1+x},$$

$$f''(x) = \frac{-1}{(1+x)^2},$$

$$f'''(x) = \frac{2}{(1+x)^3},$$

$$f^{(4)}(x) = \frac{-3 \cdot 2}{(1+x)^4},$$

\vdots

$$f^{(n)}(x) = \frac{(-1)^{n+1} (n-1)!}{(1+x)^n}$$

for $n=1, 2, 3, 4, \dots$

$$a_0 = \frac{f(0)}{0!} = \frac{\ln 1}{1} = \frac{0}{1} = 0,$$

$$a_1 = \frac{f'(0)}{1!} = \frac{1}{1} = 1,$$

$$a_2 = \frac{f''(0)}{2!} = \frac{-1}{2!} = -\frac{1}{2},$$

$$a_3 = \frac{f'''(0)}{3!} = \frac{2}{3 \cdot 2} = \frac{1}{3},$$

$$a_4 = \frac{f^{(4)}(0)}{4!} = \frac{-6}{24} = -\frac{1}{4},$$

\vdots

$$a_n = \frac{f^{(n)}(0)}{n!} = \frac{(-1)^{n+1} (n-1)!}{n!} = \frac{(-1)^{n+1}}{n}$$

$$\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \dots$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n$$

$$15.) \quad f(x) = \cos x, \quad f'(x) = -\sin x, \quad f''(x) = -\cos x, \\ f'''(x) = \sin x, \quad f^{(4)}(x) = \cos x, \quad \dots \text{ (repeat) } \dots ;$$

$$a_0 = \frac{f(0)}{0!} = \frac{\cos 0}{1} = \frac{1}{1} = 1, \quad a_1 = \frac{f'(0)}{1!} = \frac{-\sin 0}{1} = 0,$$

$$a_2 = \frac{f''(0)}{2!} = \frac{-\cos 0}{2!} = \frac{-1}{2!}, \quad a_3 = \frac{f'''(0)}{3!} = \frac{\sin 0}{3!} = 0,$$

$$a_5 = a_7 = a_9 = a_{11} = a_{\text{odd}} = 0$$

$$a_4 = \frac{f^{(4)}(0)}{4!} = \frac{\cos 0}{4!} = \frac{1}{4!}, \quad a_6 = \frac{-1}{6!}, \quad a_8 = \frac{1}{8!}, \dots;$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

$$17.) f(x) = e^{-x}, \quad f'(x) = -e^{-x}, \quad f''(x) = e^{-x}, \quad f'''(x) = -e^{-x}, \dots$$

$$a_0 = \frac{f(0)}{0!} = \frac{e^0}{1} = \frac{1}{1} = 1, \quad a_1 = \frac{f'(0)}{1!} = \frac{-e^0}{1} = -1, \quad f^{(n)}(x) = (-1)^n e^{-x}$$

$$a_2 = \frac{f''(0)}{2!} = \frac{e^0}{2!} = \frac{1}{2!}, \quad a_3 = \frac{f'''(0)}{3!} = \frac{-e^0}{3!} = \frac{-1}{3!}, \dots$$

$$a_n = \frac{f^{(n)}(0)}{n!} = \frac{(-1)^n}{n!}; \quad \text{then}$$

$$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \dots = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^n$$

$$18.) f(x) = \frac{1}{1-x} = (1-x)^{-1}$$

$$f'(x) = -(1-x)^{-2}(-1) = (1-x)^{-2}$$

$$f''(x) = -2(1-x)^{-3}(-1) = 2(1-x)^{-3}$$

$$f'''(x) = -3 \cdot 2(1-x)^{-4}(-1) = 3 \cdot 2(1-x)^{-4}$$

⋮

$$f^{(n)}(x) = n! (1-x)^{-(n+1)}$$

$$a_0 = \frac{f(0)}{0!} = \frac{1}{1} = 1,$$

$$a_1 = \frac{f'(0)}{1!} = \frac{1}{1} = 1,$$

$$a_2 = \frac{f''(0)}{2!} = \frac{2}{2} = 1,$$

$$a_3 = \frac{f'''(0)}{3!} = \frac{3 \cdot 2}{6} = 1,$$

$$a_n = \frac{f^{(n)}(0)}{n!} = \frac{n!}{n!} = 1;$$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots = \sum_{n=0}^{\infty} x^n$$

$$\begin{aligned}
 2.) \quad f(x) &= \frac{1}{1+x} = (1+x)^{-1}, & a_0 &= \frac{f(1)}{0!} = \frac{1}{2}, \\
 f'(x) &= -(1+x)^{-2}, & a_1 &= \frac{f'(1)}{1!} = \frac{-1}{2^2}, \\
 f''(x) &= 2(1+x)^{-3}, & a_2 &= \frac{f''(1)}{2!} = \frac{2}{2^3 \cdot 2} = \frac{1}{2^3}, \\
 f'''(x) &= -3 \cdot 2(1+x)^{-4}, & a_3 &= \frac{f'''(1)}{3!} = \frac{-3 \cdot 2}{2^4 \cdot 3!} = \frac{1}{2^4}, \dots \\
 &\vdots & & \\
 f^{(n)}(x) &= (-1)^n n! (1+x)^{-(n+1)}
 \end{aligned}$$

$$a_n = \frac{f^{(n)}(1)}{n!} = (-1)^n \frac{n! \cdot 2^{-(n+1)}}{n!} = \frac{(-1)^n}{2^{n+1}}; \text{ then}$$

$$\begin{aligned}
 \frac{1}{1+x} &= \frac{1}{2} - \frac{1}{2^2}(x-1) + \frac{1}{2^3}(x-1)^2 - \frac{1}{2^4}(x-1)^3 + \dots \\
 &= \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}} (x-1)^n; \text{ then}
 \end{aligned}$$

$$p_1(x;1) = \frac{1}{2} - \frac{1}{2^2}(x-1), \quad p_2(x;1) = \frac{1}{2} - \frac{1}{2^2}(x-1) + \frac{1}{2^3}(x-1)^2$$

$$\begin{aligned}
 3.) \quad p_1(x;0) &= x, \quad p_2(x;0) = x - \frac{1}{2}x^2, \\
 p_3(x;0) &= x - \frac{1}{2}x^2 + \frac{1}{3}x^3
 \end{aligned}$$

$$6.) \quad f(x) = e^x, \quad f'(x) = e^x, \quad f^{(n)}(x) = e^x \text{ for } n=0,1,2,3,\dots,$$

$$a_0 = \frac{f(2)}{0!} = \frac{e^2}{1} = e^2, \quad a_1 = \frac{f'(2)}{1!} = \frac{e^2}{1} = e^2,$$

$$a_2 = \frac{f''(2)}{2!} = \frac{e^2}{2!}, \quad a_3 = \frac{f'''(2)}{3!} = \frac{e^2}{3!}, \dots$$

$$a_n = \frac{f^{(n)}(2)}{n!} = \frac{e^2}{n!}; \text{ then}$$

$$e^x = e^2 + e^2(x-2) + \frac{e^2}{2!}(x-2)^2 + \frac{e^2}{3!}(x-2)^3 + \dots$$

$$= \sum_{n=0}^{\infty} \frac{e^2}{n!} (x-2)^n ; \text{ then}$$

$$p_1(x;2) = e^2 + e^2(x-2) , \quad p_2(x;2) = e^2 + e^2(x-2) + \frac{e^2}{2!}(x-2)^2 ,$$

$$p_3(x;2) = e^2 + e^2(x-2) + \frac{e^2}{2!}(x-2)^2 + \frac{e^2}{3!}(x-2)^3 ,$$

$$p_4(x;2) = e^2 + e^2(x-2) + \frac{e^2}{2!}(x-2)^2 + \frac{e^2}{3!}(x-2)^3 + \frac{e^2}{4!}(x-2)^4$$

$$10.) \quad p_7(x;0) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}$$

22.) The degree of $p_n(x;0)$ is $\leq n$.

$$27.) \quad f(x) = \sqrt{1+x}$$

$$f'(x) = \frac{1}{2}(1+x)^{-1/2}$$

$$f''(x) = \frac{-1}{2 \cdot 2}(1+x)^{-3/2}$$

$$f'''(x) = \frac{3}{2^3}(1+x)^{-5/2}$$

$$f^{(4)}(x) = \frac{-3 \cdot 5}{2^4}(1+x)^{-7/2}$$

⋮

$$a_0 = \frac{f(0)}{0!} = \frac{1}{1} = 1 ,$$

$$a_1 = \frac{f'(0)}{1!} = \frac{1}{2} ,$$

$$a_2 = \frac{f''(0)}{2!} = \frac{-1}{2^2 2!} ,$$

$$a_3 = \frac{f'''(0)}{3!} = \frac{3}{2^3 3!} ,$$

$$a_4 = \frac{f^{(4)}(0)}{4!} = \frac{-3 \cdot 5}{2^4 4!}$$

$$f^{(n)}(x) = (-1)^{n+1} \frac{(1 \cdot 3 \cdot 5 \cdot 7 \cdots (2n-3))}{2^n} (1+x)^{-\frac{(2n-1)}{2}}$$

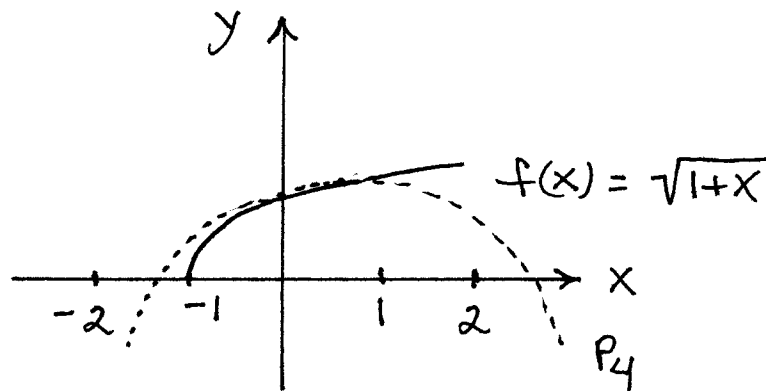
for $n = 2, 3, \dots$

$$\text{and } a_n = \frac{f^{(n)}(0)}{n!} = \frac{(-1)^{n+1} (1 \cdot 3 \cdot 5 \cdot 7 \cdots (2n-3))}{2^n n!} ,$$

$$\begin{aligned}\sqrt{1+x} &= 1 + \frac{1}{2}x - \frac{1}{2^2 2!} x^2 + \frac{3}{2^3 3!} x^3 - \dots \\ &= 1 + \frac{1}{2}x + \sum_{n=2}^{\infty} (-1)^{n+1} \frac{1 \cdot 3 \cdot 5 \cdot 7 \cdots (2n-3)}{2^n n!} x^n\end{aligned}$$

$$a.) p_4(x; 0) = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \frac{5}{128}x^4$$

b.)



$$\begin{aligned}34.) f(x) &= (1+x)^4, & f'(x) &= 4(1+x)^3, \\ f''(x) &= 12(1+x)^2, & f'''(x) &= 24(1+x), \\ f^{(4)}(x) &= 24, & f^{(5)}(x) &= 0, \dots\end{aligned}$$

$$a_0 = \frac{f(0)}{0!} = \frac{1}{1} = 1, \quad a_1 = \frac{f'(0)}{1!} = \frac{4}{1} = 4,$$

$$a_2 = \frac{f''(0)}{2!} = \frac{12}{2} = 6, \quad a_3 = \frac{f'''(0)}{3!} = \frac{24}{6} = 4,$$

$$a_4 = \frac{f^{(4)}(0)}{4!} = \frac{24}{24} = 1, \quad a_5 = a_6 = a_7 = \dots = 0;$$

then

$$(1+x)^4 = 1 + 4x + 6x^2 + 4x^3 + x^4.$$