Section 11.2

$$f(x) = \underbrace{f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n}_{\substack{\text{Taylor polynomial of degree } n, \\ P_n(x;\, a)}} + \underbrace{\frac{f^{(n+1)}(a)}{(n+1)!}(x-a)^{n+1} + \cdots}_{\substack{\text{Taylor remainder,} \\ R_n(x;\, a)}}$$

$\boxed{R_n(x; a) = \frac{f^{(n+1)}(c_n)}{(n+1)!}(x-a)^{n+1}}$, where constant c_n is between a and x. Since $f(x) = P_n(x; a) + R_n(x; a)$, a function at x $f(x)$ and its Taylor series at x $\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = \lim_{n\to\infty} P_n(x; a)$ are equal at x iff $\boxed{\lim_{n\to\infty} R_n(x; a) = 0}$.

1.) $f(x) = \cos x$, $f'(x) = -\sin x$, $f''(x) = -\cos x$, $f'''(x) = \sin x$, $f^{(iv)}(x) = \cos x$, … then

$$|f^{(n)}(x)| = \begin{cases} |\pm \cos x| \le 1 & \text{if } n = 0, 2, 4, 6, \ldots \\ |\pm \sin x| \le 1 & \text{if } n = 1, 3, 5, 7, \ldots \end{cases};$$

$$|R_n(x; 0)| = \left| \frac{f^{(n+1)}(c_n)}{(n+1)!}(x-0)^{n+1} \right| = \left| f^{(n+1)}(c_n) \right| \cdot \frac{|x|^{n+1}}{(n+1)!}$$

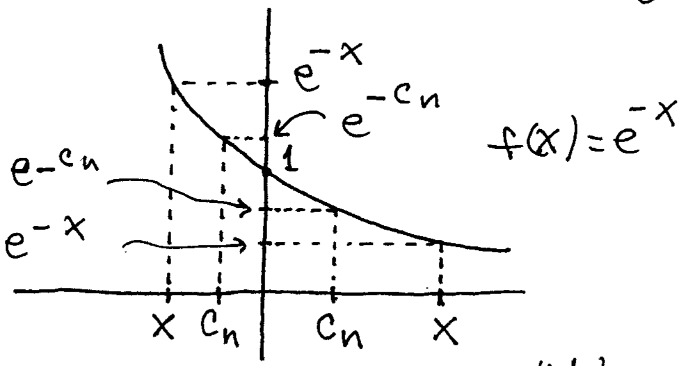
$\le 1 \cdot \frac{|x|^{n+1}}{(n+1)!}$; but $\lim_{n\to\infty} \frac{|x|^{n+1}}{(n+1)!} = 0$ (by rule) for all x-values $\Rightarrow \lim_{n\to\infty} R_n(x; 0) = 0$ for all x-values. This means that

$f(x) = \cos x$ is equal to its Maclaurin series $1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$ for all x-values.

2.) $f(x) = e^{-x}$, $f'(x) = -e^{-x}$, $f''(x) = e^{-x}$, $\ldots$, $f^{(n)}(x) = (-1)^n e^{-x}$ for $n = 0, 1, 2, 3, \ldots$; then

$$|R_n(x;0)| = \left| \frac{f^{(n+1)}(c_n)}{(n+1)!} (x-0)^{n+1} \right| = \left| (-1)^n e^{-c_n} \right| \cdot \frac{|x|^{n+1}}{(n+1)!}$$

$$= e^{-c_n} \frac{|x|^{n+1}}{(n+1)!} ,$$ where c_n is between 0 and x ;



if $0 \le c_n \le x$, then $e^{-c_n} \le 1$ and $e^{-c_n} \frac{|x|^{n+1}}{(n+1)!} \le \frac{|x|^{n+1}}{(n+1)!}$ where $\lim_{n\to\infty} \frac{|x|^{n+1}}{(n+1)!} = 0$; if $x \le c_n \le 0$, then $e^{-c_n} \le e^{-x}$ and $e^{-c_n} \frac{|x|^{n+1}}{(n+1)!} \le e^{-x} \cdot \frac{|x|^{n+1}}{(n+1)!}$ where $\lim_{n\to\infty} e^{-x} \cdot \frac{|x|^{n+1}}{(n+1)!} = e^{-x} \cdot (0) = 0$.

For both cases, $\lim_{n\to\infty} e^{-c_n} \frac{|x|^{n+1}}{(n+1)!} = 0$, so that $\lim_{n\to\infty} R_n(x;0) = 0$. This means that $f(x) = e^{-x}$ is equal to its Maclaurin series $1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots$ for all x-values

7.) $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \Rightarrow$

$e^{-1} = \cancel{1} - \cancel{1} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \frac{1}{6!} - \frac{1}{7!} + \cdots$;

$\frac{1}{6!} \approx 0.001$ and $1/7! \approx 0.0002$ so

$S_5 = \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \frac{1}{6!} \approx 0.368$ estimates the value of e^{-1} with absolute error at most 0.0002 (to 3 decimal places).

Problem A: $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \Rightarrow$

$\cos 1 = 1 - \frac{1}{2!} + \frac{1}{4!} - \frac{1}{6!} + \frac{1}{8!} - \frac{1}{10!} - \cdots$;

$1/6! \approx 0.001$ and $\frac{1}{8!} \approx 0.00002$ so

$S_4 = 1 - \frac{1}{2!} + \frac{1}{4!} - \frac{1}{6!} \approx 0.540$ estimates the value of cos 1 with absolute error at most 0.00002 (≤ 0.0001)

Problem B: $f(x) = \ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$

a.) $\ln(2) = \ln(1+1) = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots$;

$\frac{1}{n+1} \leq 0.001 \Rightarrow n+1 \geq 1000 \Rightarrow n \geq 999$, so

$S_{999} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{1}{999}$ will estimate the value of $\ln(2)$ with absolute error at most 0.001.

b.) $\ln(1.5) = \ln(1 + \frac{1}{2}) = \frac{1}{2} - \frac{1}{2 \cdot 2^2} + \frac{1}{3 \cdot 2^3} - \frac{1}{4 \cdot 2^4} + \cdots$;

$\frac{1}{7 \cdot 2^7} \approx 0.0011$ and $\frac{1}{8 \cdot 2^8} \approx 0.0005$ so

$$S_7 = \frac{1}{2} - \frac{1}{2\cdot 2^2} + \frac{1}{3\cdot 2^3} - \frac{1}{4\cdot 2^4} + \frac{1}{5\cdot 2^5} - \frac{1}{6\cdot 2^6} + \frac{1}{7\cdot 2^7} \approx 0.406$$

estimates the value of $\ln(1.5)$ with absolute error at most 0.0005 (≤ 0.001).

8.) $$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \frac{x^{n+1}}{(n+1)!} + \frac{x^{n+2}}{(n+2)!} + \cdots$$
$$= 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + R_n(x;0),$$

where $R_n(x;0) = \frac{f^{(n+1)}(c_n)}{(n+1)!}(x-0)^{n+1}$, where

$0 \leq c_n \leq x$; let $x = 2$ for e^2 then $0 \leq c_n \leq 2$

and $$R_n(2;0) = \frac{f^{(n+1)}(c_n)}{(n+1)!}2^{n+1} = \frac{e^{c_n}}{(n+1)!}2^{n+1}$$
$$\leq \frac{e^2}{(n+1)!}2^{n+1} \leq \frac{(9)2^{n+1}}{(n+1)!} \leq 0.0005 \; ?$$

n	$\frac{(9)2^{n+1}}{(n+1)!}$
8	0.013
9	0.0025
10	0.00046

Choose $n = 10$, then
$$S_{10} = 1 + 2 + \frac{2^2}{2!} + \frac{2^3}{3!} + \cdots + \frac{2^{10}}{10!}$$
≈ 7.389 estimates the value of e^2 with absolute error at most 0.00046 (≤ 0.0005) ;

calculator: $e^2 = 7.38905$

14.) $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$ then
$$\int_0^1 \frac{1 - \cos x}{x}\,dx = \int_0^1 \frac{1 - \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right)}{x}\,dx$$

$$= \int_0^1 \frac{\frac{x^2}{2!} - \frac{x^4}{4!} + \frac{x^6}{6!} - \cdots}{x}\,dx$$

$$= \int_0^1 \left(\frac{x}{2!} - \frac{x^3}{4!} + \frac{x^5}{6!} - \frac{x^7}{8!} + \cdots\right) dx$$

$$= \frac{x^2}{2\cdot 2!} - \frac{x^4}{4\cdot 4!} + \frac{x^6}{6\cdot 6!} - \frac{x^8}{8\cdot 8!} + \cdots \Big|_0^1$$

$$= \frac{1}{2\cdot 2!} - \frac{1}{4\cdot 4!} + \frac{1}{6\cdot 6!} - \frac{1}{8\cdot 8!} + \cdots$$ (an alt. series);

$\frac{1}{4\cdot 4!} \approx 0.01$ and $\frac{1}{6\cdot 6!} \approx 0.0002$, so

$S_2 = \frac{1}{2\cdot 2!} - \frac{1}{4\cdot 4!} \approx 0.2396$ estimates the value of $\int_0^1 \frac{1-\cos x}{x}\,dx$ with absolute error at most 0.0002 (to 3 decimal places)