

Section 11.5

3.) a.) $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots$; let $x = -t^2 \Rightarrow$
 $\frac{1}{1-(-t^2)} = 1 + (-t^2) + (-t^2)^2 + (-t^2)^3 + \dots \Rightarrow$
 $\frac{1}{1+t^2} = 1 - t^2 + t^4 - t^6 + t^8 - \dots$

b.) $\int_0^x \frac{1}{1+t^2} dt = \int_0^x (1 - t^2 + t^4 - t^6 + \dots) dt \Rightarrow$

$$\arctan t \Big|_0^x = \left(t - \frac{t^3}{3} + \frac{t^5}{5} - \frac{t^7}{7} + \dots \right) \Big|_0^x \Rightarrow$$

$$\arctan x - \arctan 0 = \left(x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots \right) - (0) \Rightarrow$$

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots$$

c.) $\arctan x = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{2n+1}$

d.) $\arctan\left(\frac{1}{2}\right) = \frac{1}{2} - \frac{1}{2^3 \cdot 3} + \frac{1}{2^5 \cdot 5} - \frac{1}{2^7 \cdot 7} + \frac{1}{2^9 \cdot 9} - \frac{1}{2^{11} \cdot 11} + \dots$;
 $\frac{1}{2^7 \cdot 7} \approx 0.001$ and $\frac{1}{2^9 \cdot 9} \approx 0.0002$, so

$$S_4 = \frac{1}{2} - \frac{1}{2^3 \cdot 3} + \frac{1}{2^5 \cdot 5} - \frac{1}{2^7 \cdot 7} \approx 0.4634 \text{ estimates}$$

the exact value of $\arctan\left(\frac{1}{2}\right)$ with absolute error at most 0.0002 (≤ 0.0005);
calculator: $\arctan\left(\frac{1}{2}\right) \approx 0.4636$

6.) $\frac{1 - \cos x}{1 - x^2} = \frac{1 - \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots\right)}{1 - x^2}$

$$= \frac{\frac{x^2}{2!} - \frac{x^4}{4!} + \frac{x^6}{6!} - \frac{x^8}{8!} + \dots}{1 - x^2}, \text{ so that}$$

$$\frac{x^2}{2} + \frac{11}{24}x^4 + \frac{331}{720}x^6 + \frac{3707}{8064}x^8 + \dots$$

$$1-x^2 \begin{array}{r} \frac{x^2}{2} - \frac{x^4}{24} + \frac{x^6}{720} - \frac{x^8}{40320} + \dots \\ - \left(\frac{x^2}{2} - \frac{x^4}{2} \right) \\ \hline \frac{11}{24}x^4 + \frac{x^6}{720} \\ - \left(\frac{11}{24}x^4 - \frac{11}{24}x^6 \right) \\ \hline \frac{331}{720}x^6 - \frac{x^8}{40320} \\ - \left(\frac{331}{720}x^6 - \frac{331}{720}x^8 \right) \\ \hline \frac{3707}{8064}x^8 \end{array}$$

and

$$\frac{1 - \cos x}{1 - x^2} = \frac{x^2}{2} + \frac{11}{24}x^4 + \frac{331}{720}x^6 + \frac{3707}{8064}x^8 + \dots$$

$$\begin{aligned} 7.) \quad e^x \sin x &= \left(1 + x + \frac{x^2}{2} + \dots\right) \left(x - \frac{x^3}{6} + \dots\right) \\ &= x + x^2 + \frac{x^3}{2} + \dots + \frac{-x^3}{6} - \frac{x^4}{6} - \dots \\ &= x + x^2 + \frac{1}{3}x^3 + \dots \end{aligned}$$

$$8.) \quad \frac{x}{\cos x} = \frac{x}{1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots}, \text{ so that}$$

$$1 - \frac{x^2}{2} + \frac{x^4}{24} - \dots \begin{array}{r} x + \frac{x^3}{2} + \frac{5}{24}x^5 + \dots \\ \hline \left(x - \frac{x^3}{2} + \frac{x^5}{24} - \dots \right) \end{array}$$

$$\frac{x^3}{2} - \frac{x^5}{24} + \dots$$

$$- \left(\frac{x^3}{2} - \frac{x^5}{4} + \dots \right)$$

, and

$$\frac{x}{\cos x} = x + \frac{x^3}{2} + \frac{5}{24}x^5 + \dots$$

$$\begin{aligned}
 9.) \lim_{x \rightarrow 0} \frac{1 - \cos x}{x^2} &= \lim_{x \rightarrow 0} \frac{1 - \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots\right)}{x^2} \\
 &= \lim_{x \rightarrow 0} \frac{\frac{x^2}{2!} - \frac{x^4}{4!} + \dots}{x^2} = \lim_{x \rightarrow 0} \frac{\cancel{x^2} \left(\frac{1}{2} - \frac{x^2}{4!} + \frac{x^4}{6!} - \dots\right)}{\cancel{x^2}} \\
 &= \frac{1}{2}
 \end{aligned}$$

$$\begin{aligned}
 11.) \lim_{x \rightarrow 0} \frac{\sin^2 x^3}{(1 - \cos x^2)^3} &= \lim_{x \rightarrow 0} \frac{\left(x^3 - \frac{(x^3)^3}{3!} + \frac{(x^3)^5}{5!} - \dots\right)^2}{\left(1 - \left(1 - \frac{(x^2)^2}{2!} + \frac{(x^2)^4}{4!} - \dots\right)\right)^3} \\
 &= \lim_{x \rightarrow 0} \frac{\left(x^3 - \frac{x^9}{6} + \frac{x^{15}}{120} - \dots\right)^2}{\left(\frac{x^4}{2} - \frac{x^8}{24} + \dots\right)^3} \\
 &= \lim_{x \rightarrow 0} \frac{(x^3)^2 \left(1 - \frac{x^6}{6} + \frac{x^{12}}{120} - \dots\right)^2}{(x^4)^3 \left(\frac{1}{2} - \frac{x^4}{24} + \dots\right)^3} \\
 &= \lim_{x \rightarrow 0} \frac{\left(1 - \frac{x^6}{6} + \frac{x^{12}}{120} - \dots\right)^2}{x^6 \left(\frac{1}{2} - \frac{x^4}{24} + \dots\right)^3} = \frac{1}{0^+} = \infty
 \end{aligned}$$

$$\begin{aligned}
 14.) \lim_{x \rightarrow 0} \frac{\sin x (1 - \cos x)}{e^{x^3} - 1} \\
 &= \lim_{x \rightarrow 0} \frac{\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots\right) \left(1 - \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots\right)\right)}{\left(1 + (x^3) + \frac{(x^3)^2}{2!} + \frac{(x^3)^3}{3!} + \dots\right) - 1} \\
 &= \lim_{x \rightarrow 0} \frac{x \left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \dots\right) \left(\frac{x^2}{2!} - \frac{x^4}{4!} + \dots\right)}{x^3 + \frac{x^6}{2!} + \frac{x^9}{3!} + \dots} \\
 &= \lim_{x \rightarrow 0} \frac{\cancel{x} \left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \dots\right) \cdot \cancel{x^2} \left(\frac{1}{2} - \frac{x^2}{4!} + \dots\right)}{\cancel{x^3} \left(1 + \frac{x^3}{2!} + \frac{x^6}{3!} + \dots\right)} = \frac{1}{2}
 \end{aligned}$$

$$\begin{aligned}
 15.) \text{ a.) } \int_0^1 \sqrt{x} \sin x \, dx &= \int_0^1 \sqrt{x} \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots \right) dx \\
 &= \int_0^1 \left(x^{3/2} - \frac{x^{7/2}}{3!} + \frac{x^{11/2}}{5!} - \frac{x^{15/2}}{7!} + \dots \right) dx \\
 &= \left(\frac{2}{5} x^{5/2} - \frac{2}{9} \frac{x^{9/2}}{3!} + \frac{2}{13} \frac{x^{13/2}}{5!} - \frac{2}{17} \frac{x^{17/2}}{7!} + \dots \right) \Big|_0^1 \\
 &= \left(\frac{2}{5} - \frac{2}{9 \cdot 3!} + \frac{2}{13 \cdot 5!} - \frac{2}{17 \cdot 7!} + \dots \right) - (0), \text{ and}
 \end{aligned}$$

b.) $\frac{2}{17 \cdot 7!} \approx 0.00002$ so that

$$S_3 = \frac{2}{5} - \frac{2}{9 \cdot 3!} + \frac{2}{13 \cdot 5!} \approx 0.36424 \text{ estimates}$$

the exact value of $\int_0^1 \sqrt{x} \sin x \, dx$ with absolute error at most 0.00002 (≤ 0.00005),
calculator: $\int_0^1 \sqrt{x} \sin x \, dx \approx 0.36422$.

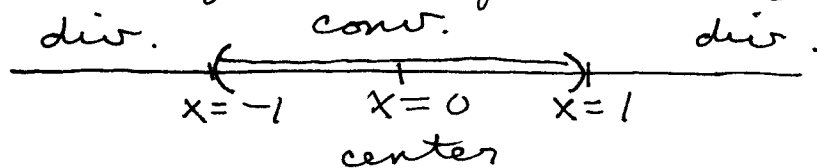
25.) a.) $\sum_{n=0}^{\infty} n^2 x^n$; $\lim_{n \rightarrow \infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n \rightarrow \infty} \frac{(n+1)^2 |x|^{n+1}}{n^2 |x|^n}$

$$= \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^2 |x| = 1 \cdot |x| = |x| < 1 \Rightarrow -1 < x < 1 ;$$

if $x=1$: $\sum_{n=0}^{\infty} n^2 (1)^n = \sum_{n=0}^{\infty} n^2$ diverges by the n th term test since $\lim_{n \rightarrow \infty} n^2 = \infty \neq 0$;

if $x=-1$: $\sum_{n=0}^{\infty} n^2 (-1)^n$ diverges by the n th term test since $\lim_{n \rightarrow \infty} n^2 (-1)^n$ DNE ($\neq 0$) ;

the interval of convergence is $\boxed{-1 < x < 1}$:



$$1.) \text{ Let } f(x) = x^2 \cdot \frac{1}{1-x} = x^2(1+x+x^2+x^3+\dots) \text{ for } |x| < 1 \\ = x^2 + x^3 + x^4 + x^5 + \dots \Rightarrow$$

$$f'(x) = \frac{(1-x)(2x) - x^2(-1)}{(1-x)^2} = \frac{2x - 2x^2 + x^2}{(1-x)^2} = \frac{2x - x^2}{(1-x)^2} \Rightarrow$$

$$\frac{2x - x^2}{(1-x)^2} = 2x + 3x^2 + 4x^3 + 5x^4 + \dots \Rightarrow$$

$$x. \frac{2x - x^2}{(1-x)^2} = \frac{2x^2 - x^3}{(1-x)^2} = 2x^2 + 3x^3 + 4x^4 + 5x^5 + \dots \Rightarrow$$

$$D \frac{2x^2 - x^3}{(1-x)^2} = \frac{(1-x)^2(4x - 3x^2) - (2x^2 - x^3) \cdot 2(1-x)(-1)}{(1-x)^4}$$

$$= \frac{x(1-x)[(1-x)(4-3x) + 2(2x-x^2)]}{(1-x)^4}$$

$$= \frac{x[4 - \cancel{4x} - 3x + 3x^2 + \cancel{4x} - 2x^2]}{(1-x)^3}$$

$$= \frac{x(x^2 - 3x + 4)}{(1-x)^3} = \frac{x^3 - 3x^2 + 4x}{(1-x)^3} \Rightarrow$$

$$\frac{x^3 - 3x^2 + 4x}{(1-x)^3} = 2^2x + 3^2x^2 + 4^2x^3 + 5^2x^4 + \dots \Rightarrow$$

$$\frac{x^4 - 3x^3 + 4x^2}{(1-x)^3} = 2^2x^2 + 3^2x^3 + 4^2x^4 + 5^2x^5 + \dots$$

c.) The formula will give the correct answer for $x = \frac{1}{3}$ since the interval of convergence is $-1 < x < 1$.

Chapter 11 Review Exercises

18.) $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$ so that
 $e^{1/10} \approx 1 + \frac{1}{10} + \frac{1}{10^2 \cdot 2!} = 1.105$; the error
 $R_2\left(\frac{1}{10}; 0\right) = \frac{f^{(3)}(c_n) \left(\frac{1}{10} - 0\right)^3}{3!}$, where $0 \leq c_n \leq \frac{1}{10}$,
 $= \frac{e^{c_n}}{10^3 \cdot 3!} < \frac{e^{1/10}}{10^3 \cdot 3!} < \frac{2}{10^3 \cdot 3!} \approx 0.0003$.

19.) $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$ so that
 $\cos\left(\frac{1}{3}\right) \approx 1 - \frac{1}{3^2 \cdot 2!} + \frac{1}{3^4 \cdot 4!} \approx 0.945$; the absolute
error $|R_3| < \frac{1}{3^6 \cdot 6!} \approx 0.000002$

33.) a.) $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots \Rightarrow$
 $\cos x^3 = 1 - \frac{(x^3)^2}{2!} + \frac{(x^3)^4}{4!} - \frac{(x^3)^6}{6!} + \dots$
 $= 1 - \frac{x^6}{2!} + \frac{x^{12}}{4!} - \frac{x^{18}}{6!} + \dots \Rightarrow$
 $\int_0^{\frac{1}{2}} \cos x^3 dx = \int_0^{\frac{1}{2}} \left(1 - \frac{x^6}{2!} + \frac{x^{12}}{4!} - \frac{x^{18}}{6!} + \dots\right) dx$
 $= \left(x - \frac{x^7}{7 \cdot 2!} + \frac{x^{13}}{13 \cdot 4!} - \frac{x^{19}}{19 \cdot 6!} + \dots\right) \Big|_0^{\frac{1}{2}}$
 $\approx \frac{1}{2} - \frac{1}{2^7 \cdot 7 \cdot 2!} + \frac{1}{2^{13} \cdot 13 \cdot 4!} \approx 0.4994$; the absolute
error $|R_3| < \frac{1}{2^{19} \cdot 19 \cdot 6!} \approx 0.0000000001$.

d.) $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$ so that
 $e^{-x^3} = 1 + (-x^3) + \frac{(-x^3)^2}{2!} + \frac{(-x^3)^3}{3!} + \dots$

$$\begin{aligned}
&= 1 - x^3 + \frac{x^6}{2!} - \frac{x^9}{3!} + \dots \Rightarrow \\
\int_1^2 e^{-x^3} dx &= \int_1^2 \left(1 - x^3 + \frac{x^6}{2!} - \frac{x^9}{3!} + \dots \right) dx \\
&= \left(x - \frac{x^4}{4} + \frac{x^7}{7 \cdot 2!} - \frac{x^{10}}{10 \cdot 3!} + \dots \right) \Big|_1^2 \\
&= \left(2 - \frac{2^4}{4} + \frac{2^7}{7 \cdot 2!} - \frac{2^{10}}{10 \cdot 3!} + \dots \right) - \left(1 - \frac{1}{4} + \frac{1}{7 \cdot 2!} - \frac{1}{10 \cdot 3!} + \dots \right) \\
&\approx \left(2 - \frac{2^4}{4} + \frac{2^7}{7 \cdot 2!} \right) - \left(1 - \frac{1}{4} + \frac{1}{7 \cdot 2!} \right) \approx 6.321 ; \text{ the}
\end{aligned}$$

errors for the 2 alternating series are
 $|R_3| < \frac{2^{10}}{10 \cdot 3!} \approx 17.1$ and $|R_3| < \frac{1}{10 \cdot 3!} \approx 0.017$ so
that the total absolute error is
 $|R_3| \leq 17.1 + 0.017 = 17.117$

calculator : $\int_1^2 e^{-x^3} dx \approx 0.085 !$

The estimate is a poor one because the interval $[1, 2]$ is "far" from the center ($x=0$) of the Maclaurin series.

$$49.) \quad \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \dots \Rightarrow$$

$$\cos x^2 = 1 - \frac{x^4}{2!} + \frac{x^8}{4!} - \dots ; \text{ then}$$

$$\lim_{x \rightarrow 0} \frac{(1 - \cos x^2)^5}{(x - \sin x)^{20}} = \lim_{x \rightarrow 0} \frac{\left(\frac{x^4}{2!} - \frac{x^8}{4!} + \frac{x^{12}}{6!} - \dots \right)^5}{\left(x - \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots \right) \right)^{20}}$$

$$= \lim_{x \rightarrow 0} \frac{\left(x^4 \left(\frac{1}{2!} - \frac{x^4}{4!} + \frac{x^8}{6!} - \dots \right) \right)^5}{\left(x^3 \left(\frac{1}{3!} - \frac{x^2}{5!} + \frac{x^4}{7!} - \dots \right) \right)^{20}}$$

$$\begin{aligned}
&= \lim_{x \rightarrow 0} \frac{x^{20} \left(\frac{1}{2!} - \frac{x^4}{4!} + \frac{x^8}{6!} - \dots \right)^5}{x^{60} \left(\frac{1}{3!} - \frac{x^2}{5!} + \frac{x^4}{7!} - \dots \right)^{20}} \\
&= \lim_{x \rightarrow 0} \frac{1}{x^{40}} \cdot \frac{\left(\frac{1}{2!} - \frac{x^4}{4!} + \frac{x^8}{6!} - \dots \right)^5}{\left(\frac{1}{3!} - \frac{x^2}{5!} + \frac{x^4}{7!} - \dots \right)^{20}} \\
&= \frac{1}{0^+} \cdot \frac{\left(\frac{1}{2} \right)^5}{\left(\frac{1}{6} \right)^{20}} = +\infty.
\end{aligned}$$

$$\begin{aligned}
51.) \quad \sum_{n=0}^{\infty} 2^n x^n &= \sum_{n=0}^{\infty} (2x)^n = 1 + (2x) + (2x)^2 + (2x)^3 + \dots \\
&= \frac{1}{1-(2x)} \Rightarrow f(x) = \frac{1}{1-2x}; \text{ then}
\end{aligned}$$

$$f'(x) = 0(1-2x)^{-1} = -1 \cdot (1-2x)^{-2} (-2) = 2(1-2x)^{-2},$$

$$f''(x) = -2^2(1-2x)^{-3} (-2) = 2^2 \cdot 2(1-2x)^{-3},$$

$$f'''(x) = -2^2 \cdot 3 \cdot 2(1-2x)^{-4} \cdot (-2) = 2^3 3! (1-2x)^{-4},$$

$$f^{(4)}(x) = -2^3 4! (1-2x)^{-5} (-2) = 2^4 4! (1-2x)^{-5}, \dots,$$

$$f^{(n)}(x) = 2^n n! (1-2x)^{-(n+1)} \quad \text{so that}$$

$$f^{(33)}(0) = 2^{33} 33! (1)^{-34} = 2^{33} \cdot 33!$$