Section 11.6

1.) a.) $(2+3i)+(5-2i) = 7+i$

d.) $\dfrac{3+2i}{4-i} = \dfrac{3+2i}{4-i}\cdot\dfrac{4+i}{4+i} = \dfrac{12+11i+2i^2}{16-i^2}$

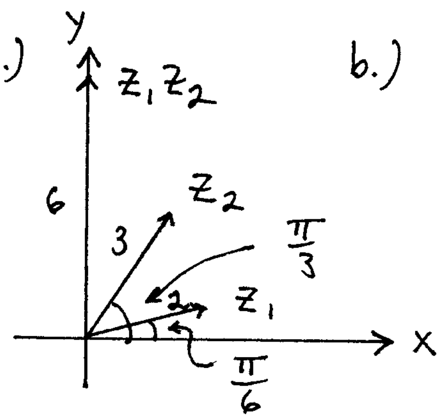
$= \dfrac{12+11i-2}{16-(-1)} = \dfrac{10}{17} + \dfrac{11}{17}i$

2.) a.) $(2+3i)^2 = 4+12i+9i^2 = 4+12i-9 = -5+12i$

d.) $\dfrac{1+5i}{2-3i} = \dfrac{1+5i}{2-3i}\cdot\dfrac{2+3i}{2+3i} = \dfrac{2+13i+15i^2}{4-9i^2}$

$= \dfrac{2+13i-15}{4-9(-1)} = \dfrac{-13}{13}+\dfrac{13i}{13} = -1+i$

3.) a.)



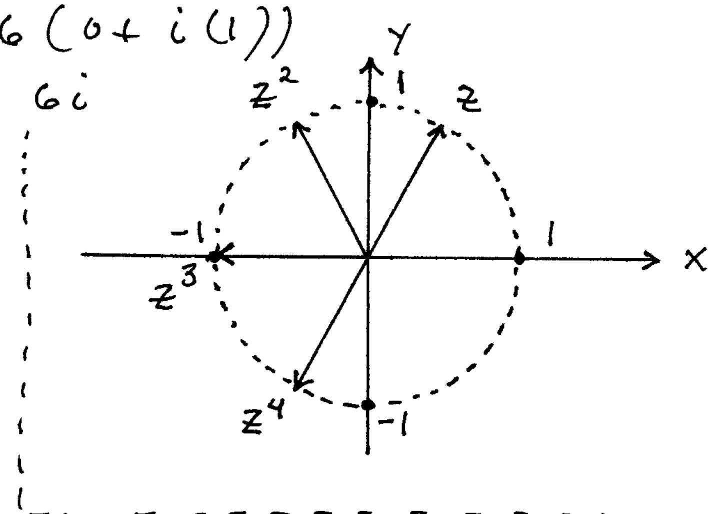
b.) $z_1 z_2 = 2\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)\cdot 3\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)$

$= 6\left(\cos\left(\frac{\pi}{6}+\frac{\pi}{3}\right)+i\sin\left(\frac{\pi}{6}+\frac{\pi}{3}\right)\right)$

$= 6\left(\cos\left(\frac{\pi}{2}\right)+i\sin\left(\frac{\pi}{2}\right)\right)$

$= 6(0+i(1))$

$= 6i$



5.) $z = 1\cdot\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)$,

$z^2 = \left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)^2$

$= \cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}$,

$z^3 = \left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)^3$

$= \cos\frac{3\pi}{3}+i\sin\frac{3\pi}{3} = \cos\pi + i\sin\pi$,

$z^4 = \left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)^4$

$= \cos\frac{4\pi}{3}+i\sin\frac{4\pi}{3}$;

6.) a.) $i^3 = i^2 \cdot i = -i$

b.) $i^4 = i^2 \cdot i^2 = (-1)(-1) = 1$

c.) $i^5 = i^4 \cdot i = (1)i = i$

d.) $i^{73} = (i^4)^{18} \cdot i = (1)^{18} \cdot i = i$

7.) $z = 2(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6})$,

a.) $z^2 = 2^2(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6})^2$

$= 4\ (\cos\frac{2\pi}{6} + i\sin\frac{2\pi}{6}) = 4\,(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3})$,

mag $(z^2) = 4$ and arg $(z^2) = \pi/3$

b.) $z^3 = 2^3(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6})^3$

$= 8\ (\cos\frac{3\pi}{6} + i\sin\frac{3\pi}{6}) = 8\,(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2})$,

mag $(z^3) = 8$ and arg $(z^3) = \frac{\pi}{2}$

c.) $z^4 = 2^4(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6})^4$

$= 16\,(\cos\frac{4\pi}{6} + i\sin\frac{4\pi}{6}) = 16\,(\cos\frac{2}{3}\pi + i\sin\frac{2}{3}\pi)$,

mag $(z^4) = 16$ and arg $(z^4) = \frac{2}{3}\pi$

d.) $z^n = 2^n(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6})^n$

$= 2^n\,(\cos\frac{n\pi}{6} + i\sin\frac{n\pi}{6})$,

mag $(z^n) = 2^n$ and arg $(z^n) = \frac{n\pi}{6}$

11.) $z = r(\cos\theta + i\sin\theta)$ and $w = \frac{1}{r}(\cos(-\theta) + i\sin(-\theta))$

then $zw = r(\cos\theta + i\sin\theta) \cdot \frac{1}{r}(\cos(-\theta) + i\sin(-\theta))$

$= \frac{r}{r}(\cos(\theta - \theta) + i\sin(\theta - \theta))$

$= 1(\cos 0 + i\sin 0) = 1(1 + i(0)) = 1$.

12.) $z = 4+4i \Rightarrow |z| = \sqrt{4^2+4^2} = \sqrt{32} = 4\sqrt{2} \Rightarrow$

$z = 4\sqrt{2}\left(\frac{4}{4\sqrt{2}} + \frac{4}{4\sqrt{2}} i\right) = 4\sqrt{2}\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} i\right)$

$= 4\sqrt{2}\left(\cos\frac{\pi}{4} + i \sin\frac{\pi}{4}\right)$; by 11.)

$z^{-1} = \frac{1}{4\sqrt{2}}\left(\cos\left(\frac{-\pi}{4}\right) + i \sin\left(\frac{-\pi}{4}\right)\right)$

$= \frac{1}{4\sqrt{2}}\left(\frac{+1}{\sqrt{2}} + i\left(\frac{-1}{\sqrt{2}}\right)\right) = \frac{1}{8} - \frac{i}{8}$

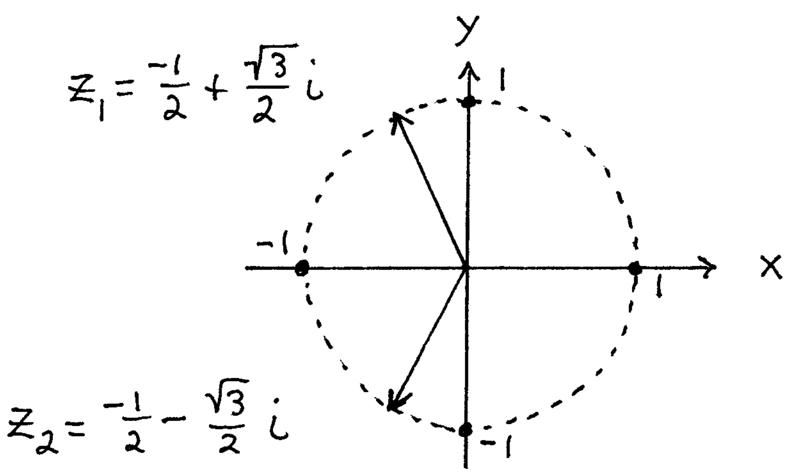
13.) a.) $z = 2+3i$ and $z^2 - 4z + 13 = (2+3i)^2 - 4(2+3i) + 13$

$= 4 + 12i + 9i^2 - 8 - 12i + 13 = 9 - 9 + 0 = 0$

b.) $z^2 - 4z + 13 = 0 \Rightarrow$

$$z = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(13)}}{2} = \frac{4 \pm \sqrt{-36}}{2} = 2 \pm 3i$$

14.) a.) $z^2 + z + 1 = 0 \Rightarrow z = \frac{-1 \pm \sqrt{1-4}}{2} = \frac{-1}{2} \pm \frac{\sqrt{3}}{2} i$



19.) d.) $3(\cos 42° + i \sin 42°) \cdot 5(\cos 168° + i \sin 168°)$

$= 15(\cos(42° + 168°) + i \sin(42° + 168°))$

$= 15(\cos 210° + i \sin 210°)$

$= 15\left(-\sqrt{3}/2 + i\left(\frac{-1}{2}\right)\right) = -\frac{15\sqrt{3}}{2} - \frac{15}{2} i$

e.) $\dfrac{\sqrt{8}\,(\cos 147^\circ + i\sin 147^\circ)}{\sqrt{2}\,(\cos 57^\circ + i\sin 57^\circ)}$

$= \dfrac{\sqrt{8}}{\sqrt{2}}\left(\cos(147^\circ - 57^\circ) + i\sin(147^\circ - 57^\circ)\right)$

$= 2(\cos 90^\circ + i\sin 90^\circ) = 2(0 + i(1)) = 2i$

g.) (by problem 11.)

$[3(\cos 52^\circ + i\sin 52^\circ)]^{-1}$

$= \frac{1}{3}\left(\cos(-52^\circ) + i\sin(-52^\circ)\right)$

$= \frac{1}{3}(\cos 52^\circ - i\sin 52^\circ)$

h.) $\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)^{12} = \cos\left(12\frac{\pi}{6}\right) + i\sin\left(12\frac{\pi}{6}\right)$

$= \cos 2\pi + i\sin 2\pi = 1 + i(0) = 1$

20.) d.) $\left(\cos\frac{\pi}{12} + i\sin\frac{\pi}{12}\right)^{20} = \cos\left(\frac{20\pi}{12}\right) + i\sin\left(\frac{20\pi}{12}\right)$

$= \cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3} = \frac{1}{2} + i\left(\frac{-\sqrt{3}}{2}\right) = \frac{1}{2} - \frac{\sqrt{3}}{2}i$

23.) <u>FACT</u>: $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta \Rightarrow$

$(\cos\theta + i\sin\theta)^3 = \cos 3\theta + i\sin 3\theta \Rightarrow$

$\cos^3\theta + 3\cos^2\theta\cdot(\sin\theta\cdot i) + 3\cos\theta\cdot(\sin\theta\cdot i)^2$

$+ (i\sin\theta)^3 = \cos 3\theta + i\sin 3\theta \Rightarrow$

$\cos^3\theta + (3\sin\theta\cos^2\theta)i - 3\sin^2\theta\cos\theta$

$- \sin^3\theta\cdot i = \cos 3\theta + i\sin 3\theta \Rightarrow$

$(\cos^3\theta - 3\sin^2\theta\cos\theta) + (3\sin\theta\cos^2\theta - \sin^3\theta)i$

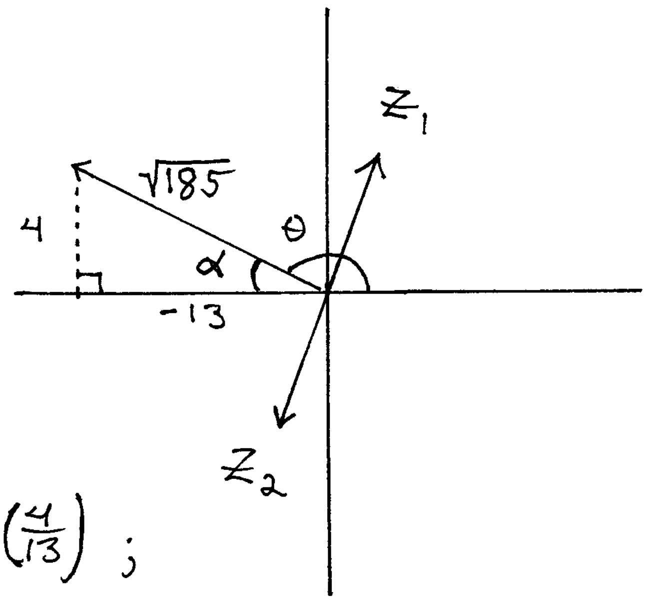
$= \cos 3\theta + i\sin 3\theta \Rightarrow$

$$\cos 3\theta = \cos^3\theta - 3\sin^2\theta\cos\theta \quad \text{and}$$
$$\sin 3\theta = 3\sin\theta\cos^2\theta - \sin^3\theta \;.$$

36.) $x^2 + (i)x + 3 - i = 0 \;\Rightarrow$

$$x = \frac{-i \pm \sqrt{(i)^2 - 4(1)(3-i)}}{2(1)} = \frac{-i \pm \sqrt{-1-12+4i}}{2}$$
$$= \frac{-i \pm \sqrt{-13+4i}}{2} \;;$$

$z = -13 + 4i$



$\tan\alpha = \dfrac{4}{13}$

so $\alpha = \arctan\left(\dfrac{4}{13}\right)$

and $\theta = \pi - \alpha$

$\quad = \pi - \arctan\left(\dfrac{4}{13}\right)$;

then

$$z_1 = 185^{1/4}\left(\cos\left(\frac{\theta}{2}\right) + i\sin\left(\frac{\theta}{2}\right)\right) ,$$
$$z_2 = 185^{\frac{1}{4}}\left(\cos\left(\frac{\theta}{2}+\pi\right) + i\sin\left(\frac{\theta}{2}+\pi\right)\right) , \quad \text{and}$$

$$x = \frac{-i + z_1}{2} \quad \text{or} \quad x = \frac{-i + z_2}{2} \;\Rightarrow$$

$$x = \frac{-i \pm z_1}{2} = \frac{1}{2}\left(-i \pm 185^{1/4}\cos\left(\frac{\theta}{2}\right) \pm 185^{\frac{1}{4}}\sin\left(\frac{\theta}{2}\right)i\right) \Rightarrow$$

$$x = \pm\frac{185^{1/4}}{2}\cos\left(\frac{\theta}{2}\right) + i\left(\frac{-1}{2} \pm \frac{185^{\frac{1}{4}}}{2}\sin\left(\frac{\theta}{2}\right)\right) ,$$

where $\theta = \pi - \arctan\left(\dfrac{4}{13}\right)$.

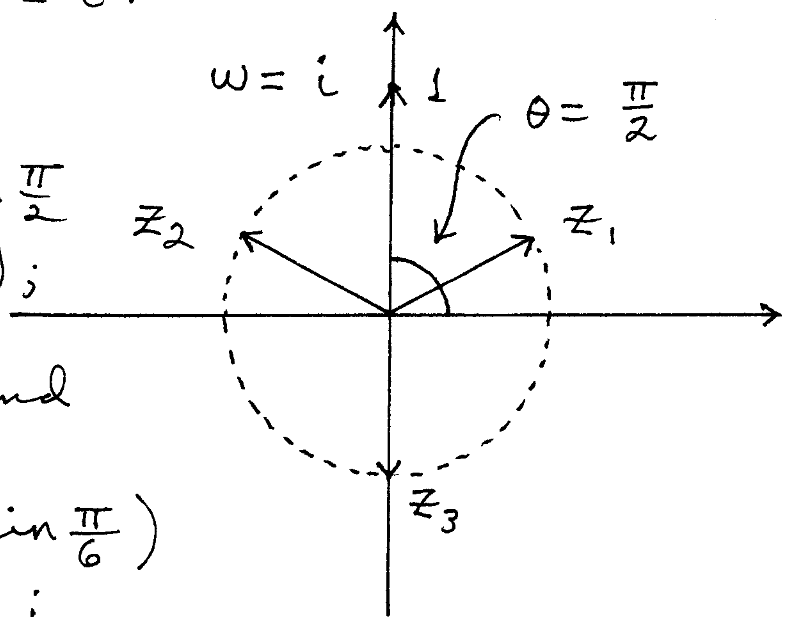
21.) Solve $z^3 = i$:

$w = i$

$= 0 + i \cdot 1$

$= \cos\frac{\pi}{2} + i\sin\frac{\pi}{2}$

$= 1\cdot\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$;



then $\frac{\theta}{3} = \frac{\pi}{6}$ and

$z_1 = 1\cdot\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$

$= \frac{\sqrt{3}}{2} + \frac{1}{2}i$,

$z_2 = 1\cdot\left(\cos\left(\frac{\pi}{6} + \frac{2\pi}{3}\right) + i\sin\left(\frac{\pi}{6} + \frac{2\pi}{3}\right)\right)$

$= \cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}$

$= \frac{-\sqrt{3}}{2} + \frac{1}{2}i$,

$z_3 = 1\cdot\left(\cos\left(\frac{\pi}{6} + \frac{4\pi}{3}\right) + i\sin\left(\frac{\pi}{6} + \frac{4\pi}{3}\right)\right)$

$= \cos\frac{9\pi}{6} + i\sin\frac{9\pi}{6}$

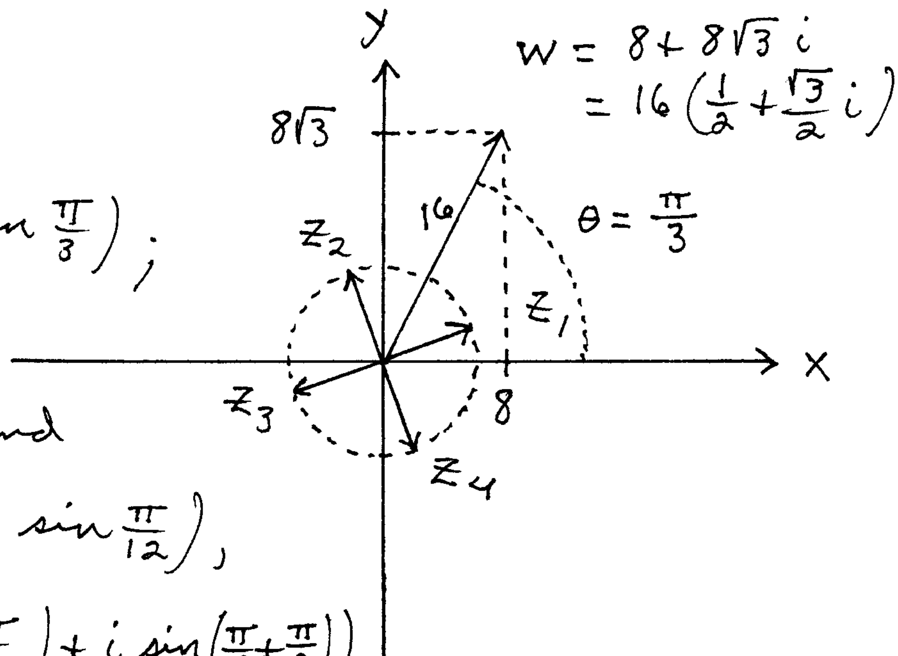
$= 0 + i(-1) = -i$.

10.) Solve $z^4 = 8 + 8\sqrt{3}\, i$:

$w = 8 + 8\sqrt{3}\, i$

$= 16\left(\frac{1}{2} + \frac{\sqrt{3}}{2} i\right)$

$= 16\left(\cos\frac{\pi}{3} + i \sin\frac{\pi}{3}\right)$;



then $\frac{\theta}{4} = \frac{\pi}{12}$ and

$z_1 = 2\left(\cos\frac{\pi}{12} + i \sin\frac{\pi}{12}\right)$,

$z_2 = 2\left(\cos\left(\frac{\pi}{12} + \frac{\pi}{2}\right) + i \sin\left(\frac{\pi}{12} + \frac{\pi}{2}\right)\right)$

$= 2\left(\cos\frac{7\pi}{12} + i \sin\frac{7\pi}{12}\right)$,

$z_3 = 2\left(\cos\left(\frac{\pi}{12} + \pi\right) + i \sin\left(\frac{\pi}{12} + \pi\right)\right)$

$= 2\left(\cos\frac{13\pi}{2} + i \sin\frac{13\pi}{2}\right)$,

$z_4 = 2\left(\cos\left(\frac{\pi}{12} + \frac{3\pi}{2}\right) + i \sin\left(\frac{\pi}{12} + \frac{3\pi}{2}\right)\right)$

$= 2\left(\cos\frac{19\pi}{2} + i \sin\frac{19\pi}{2}\right)$