Section 14.4 Solutions

1.) $\displaystyle\lim_{(x,y)\to(2,3)} \frac{x+y}{x^2+y^2} = \frac{5}{4+9} = \frac{5}{13}$.

2.) $\displaystyle\lim_{(x,y)\to(1,1)} \frac{x^2}{x^2+y^2} = \frac{1}{1+1} = \frac{1}{2}$.

3.) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2}{x^2+y^2} = \text{"}\frac{0}{0}\text{"}\ ?$;

check paths through $(0,0)$:

line $y=0$: $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2}{x^2+0} = \lim_{x\to 0} 1 = 1$;

line $y=x$: $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2}{x^2+x^2} = \lim_{x\to 0} \frac{x^2}{2x^2} = \lim_{x\to 0} \frac{1}{2} = \frac{1}{2}$;

we can conclude that $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2}{x^2+y^2}$ does not exist (DNE).

4.) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2} = \text{"}\frac{0}{0}\text{"}\ ?$;

check paths through $(0,0)$:

line $y=0$: $\displaystyle\lim_{(x,y)\to(0,0)} \frac{0}{x^2} = \lim_{x\to 0} 0 = 0$;

line $y=x$: $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2}{x^2+x^2} = \lim_{x\to 0} \frac{1}{2} =$;

we can conclude that $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$ DNE.

5.) $\lim\limits_{(x,y)\to(2,3)} x^y = 2^3 = 8.$

6.) $\lim\limits_{(x,y)\to(0,0)} (x^2)^y = \text{"}0^0\text{"}\ ?$;

<u>check paths through $(0,0)$</u>:

<u>line $y=0$</u>: $\lim\limits_{(x,y)\to(0,0)} (x^2)^0 = \lim\limits_{x\to 0} 1 = 1$;

<u>line $x=0$</u>: $\lim\limits_{(x,y)\to(0,0)} (0)^y = \lim\limits_{y\to 0} 0 = 0$;

we can conclude that $\lim\limits_{(x,y)\to(0,0)} (x^2)^y$ DNE.

7.) $\lim\limits_{(x,y)\to(0,0)} (1+xy)^{1/xy} = \lim\limits_{w\to 0} (1+w)^{1/w} = e.$

8.) $\lim\limits_{(x,y)\to(0,0)} (1+x)^{1/y} = \text{"}1^{\pm\infty}\text{"}\ ?$;

<u>check paths through $(0,0)$</u>:

<u>line $y=x$</u>: $\lim\limits_{(x,y)\to(0,0)} (1+x)^{1/x} = \lim\limits_{x\to 0} (1+x)^{1/x} = e$;

<u>line $x=0$</u>: $\lim\limits_{(x,y)\to(0,0)} (1+0)^{1/y} = \lim\limits_{y\to 0} 1^{\frac{1}{y}} = \lim\limits_{y\to 0} 1 = 1$;

we can conclude that $\lim\limits_{(x,y)\to(0,0)} (1+x)^{1/y}$ DNE.

9.) a.) $f(x,y) = \dfrac{1}{x+y}$; Domain is all points

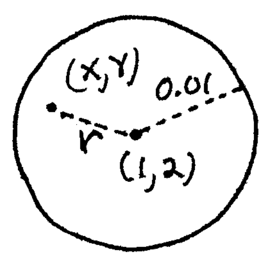
(x,y) for which $x+y \neq 0$, i.e., $y \neq -x$.

12.) a.) $f(x,y) = \sqrt{x^2+y^2-25}$; Domain is all points (x,y) for which $x^2+y^2-25 \geq 0$, i.e., $x^2+y^2 \geq 5^2$, i.e., all points lying <u>on</u> or <u>outside</u> the circle $x^2+y^2=5^2$.

14.) a.) $f(x,y) = \dfrac{1}{\sqrt{49-x^2-y^2}}$; Domain is all points (x,y) for which $49-x^2-y^2>0$, i.e., $x^2+y^2<7^2$, i.e., all points lying <u>inside</u> the circle $x^2+y^2=7^2$.

21.) $f(x,y) = x+y$

a.) Distance $r = \sqrt{(x-1)^2+(y-2)^2} < 0.01$;



thus

$|x-1| = \sqrt{(x-1)^2} \leq \sqrt{(x-1)^2+(y-2)^2}$

$\rightarrow |x-1| < 0.01$ and

$|y-2| = \sqrt{(y-2)^2} \leq \sqrt{(x-1)^2+(y-2)^2} \rightarrow |y-2|<0.01$.

b.) assume $|x-1|<0.01$ and $|y-2|<0.01 \rightarrow$

$-0.01 < x-1 < 0.01$ and $-0.01 < y-2 < 0.01 \rightarrow$

$\underline{0.99 < x < 1.01}$ and $\underline{1.99 < y < 2.01}$;

show that $|f(x,y)-3| < 0.02$; then

$|f(x,y)-3| = |(x+y)-3|$, but

$0.99 + 1.99 < (X+Y) < 1.01 + 2.01 \rightarrow$

$2.98 < (X+Y) < 3.02$ so that

$|f(X,Y) - 3| = |(X+Y) - 3| < 0.02$

(OR USE TRIANGLE INEQUALITY :

$$
\begin{aligned}
|f(X,Y)-3| &= |(X+Y)-3| \\
&= |X-1+Y-2| \\
&\underset{\text{triangle inequality}}{\le} |X-1| + |Y-2| \\
&< 0.01 + 0.01 = 0.02 .)
\end{aligned}
$$

c.) Find $\delta > 0$ so that

if $0 < \sqrt{(X-1)^2 + (Y-2)^2} < \delta$, then $|f(X,Y) - 3| < 0.001$:

Let $\delta = \frac{1}{2}(0.001) = 0.0005$. Then

$|X-1| \le \sqrt{(X-1)^2 + (Y-2)^2} < 0.0005$ and

$|Y-2| \le \sqrt{(X-1)^2 + (Y-2)^2} < 0.0005$, so that

$$
\begin{aligned}
|f(X,Y) - 3| &= |(X+Y) - 3| \\
&= |X-1+Y-2| \\
&\le |X-1| + |Y-2| \\
&< 0.0005 + 0.0005 = 0.001 .
\end{aligned}
$$

d.) Find $\delta > 0$ so that

if $0 < \sqrt{(X-1)^2 + (Y-2)^2} < \delta$, then $|f(X,Y) - 3| < \varepsilon$:

Let $\delta = \frac{1}{2}\varepsilon$. Then

$$|x-1| \le \sqrt{(x-1)^2+(y-2)^2} < \tfrac{1}{2}\varepsilon \quad \text{and}$$
$$|y-2| \le \sqrt{(x-1)^2+(y-2)^2} < \tfrac{1}{2}\varepsilon \text{ , so that}$$

$$\begin{aligned} |f(x,y)-3| &= |(x+y)-3| \\ &= |x-1+y-2| \\ &\le |x-1|+|y-2| \\ &< \tfrac{1}{2}\varepsilon + \tfrac{1}{2}\varepsilon = \varepsilon . \end{aligned}$$

e.) $\displaystyle\lim_{(x,y)\to(1,2)} (x+y) = 3$

24.) $f(x,y) = \dfrac{5x^2y}{2x^4+3y^2}$

a.) Domain : all points (x,y) except $(0,0)$.

b.) line $y=2x$: $\displaystyle\lim_{(x,y)\to(0,0)} \frac{5x^2(2x)}{2x^4+3(2x)^2} = \lim_{x\to 0}\frac{10x^3}{2x^4+12x^2}$

$$= \lim_{x\to 0}\frac{\cancel{2x^2}(5x)}{\cancel{2x^2}(x^2+6)} = \lim_{x\to 0}\frac{5x}{x^2+6} = \frac{0}{0+6} = 0 .$$

c.) line $y=3x$: $\displaystyle\lim_{(x,y)\to(0,0)} \frac{5x^2(3x)}{2x^4+3(3x)^2} = \lim_{x\to 0}\frac{15x^3}{2x^4+27x^2}$

$$= \lim_{x\to 0}\frac{\cancel{x^2}(15x)}{\cancel{x^2}(2x^2+27)} = \lim_{x\to 0}\frac{15x}{2x^2+27} = \frac{0}{0+27} = 0 .$$

d.) path $y=x^2$: $\displaystyle\lim_{(x,y)\to(0,0)} \frac{5x^2(x^2)}{2x^4+3(x^2)^2} = \lim_{x\to 0}\frac{5x^4}{2x^4+3x^4}$

$$= \lim_{x\to 0}\frac{5x^4}{5x^4} = \lim_{x\to 0} 1 = 1 .$$

e.) Conclusion : $\displaystyle\lim_{(x,y)\to(0,0)} \frac{5x^2y}{2x^4+3y^2}$ DNE .