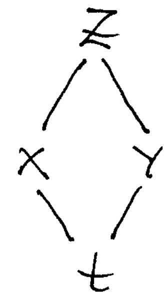
Section 14.6 Solutions



1.) $z = x^2y^3,\ x = t^2,\ y = t^3$

a.) $\dfrac{dz}{dt} = z_x \cdot \dfrac{dx}{dt} + z_y \cdot \dfrac{dy}{dt}$

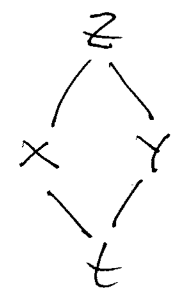
$= (2xy^3)(2t) + (3x^2y^2)(3t^2)$

$= 4(t^2)(t^3)^3(t) + 9(t^2)^2(t^3)^2(t^2)$

$= 4t^{12} + 9t^{12} = 13t^{12}.$

b.) $z = x^2y^3 = (t^2)^2(t^3)^3 = t^{13} \rightarrow$

$\dfrac{dz}{dt} = 13t^{12}.$



2.) $z = xe^y,\ x = t,\ y = 1+3t$

a.) $\dfrac{dz}{dt} = z_x \cdot \dfrac{dx}{dt} + z_y \cdot \dfrac{dy}{dt}$

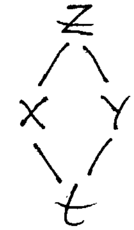
$= e^y \cdot (1) + xe^y \cdot (3)$

$= e^{1+3t} + 3te^{1+3t} = (1+3t)e^{1+3t}.$

b.) $z = xe^y = te^{1+3t} \longrightarrow$

$\dfrac{dz}{dt} = t \cdot e^{1+3t}(3) + (1) \cdot e^{1+3t} = (1+3t)e^{1+3t}$

4.) a.) $z = \ln(x+3y),\ x = t^2,\ y = \tan(3t) \rightarrow$

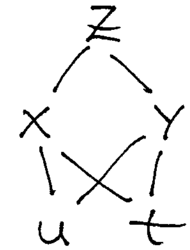


$\dfrac{dz}{dt} = z_x \cdot \dfrac{dx}{dt} + z_y \cdot \dfrac{dy}{dt}$

$= \dfrac{1}{x+3y} \cdot (2t) + \dfrac{3}{x+3y} \cdot \sec^2(3t) \cdot 3$

$= \dfrac{2t}{t^2 + 3\tan(3t)} + \dfrac{9\sec^2(3t)}{t^2+3\tan(3t)} = \dfrac{2t + 9\sec^2(3t)}{t^2 + 3\tan(3t)}$

5.) a.) $z = x^2 y, \; x = 3t + 4u, \; y = 5t - u \rightarrow$



$$\frac{\partial z}{\partial t} = z_x \cdot \frac{\partial x}{\partial t} + z_y \cdot \frac{\partial y}{\partial t}$$

$$= (2xy)(3) + (x^2)(5)$$

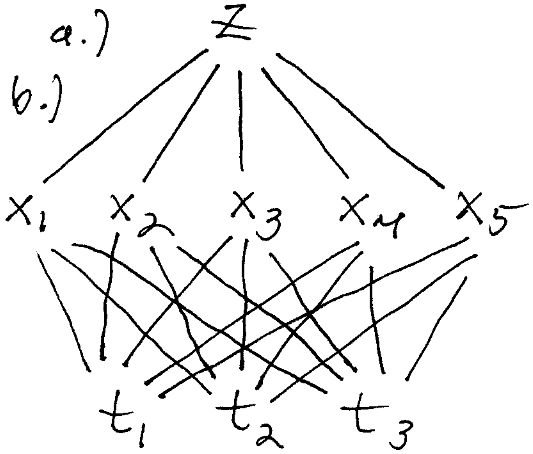
$$= 2(3t+4u)(5t-u)(3) + 5(3t+4u)^2$$

$$= (3t+4u)\left[(30t-6u) + 5(3t+4u)\right]$$

$$= (3t+4u)\left[45t + 14u\right]$$

$$= 135t^2 + 222ut + 56u^2$$

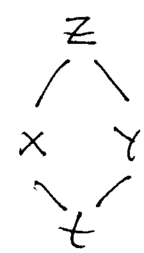
7.) a.)
b.)



c.) $$\frac{\partial z}{\partial t_3} = z_{x_1} \cdot \frac{\partial x_1}{\partial t_3} + z_{x_2} \cdot \frac{\partial x_2}{\partial t_3} + z_{x_3} \cdot \frac{\partial x_3}{\partial t_3} + z_{x_4} \cdot \frac{\partial x_4}{\partial t_3} + z_{x_5} \cdot \frac{\partial x_5}{\partial t_3}$$

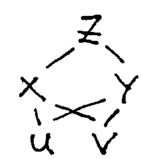
d.) x_1, x_3, x_4, x_5 e.) t_1 and t_2

10.) $$\frac{dz}{dt} = z_x \cdot \frac{\partial x}{\partial t} + z_y \cdot \frac{\partial y}{dt}$$



$$= (3)(4) + (2)(-3) = 6$$

12.) $z = f(x,y), \; x = u^2 - v^2, \; y = v^2 - u^2$



a.) $$\frac{\partial z}{\partial v} = f_x \cdot \frac{\partial x}{\partial v} + f_y \cdot \frac{\partial y}{\partial v} = f_x(-2v) + f_y(2v)$$

$$\frac{\partial z}{\partial u} = f_x \cdot \frac{\partial x}{\partial u} + f_y \cdot \frac{\partial y}{\partial u} = f_x(2u) + f_y(-2u) \rightarrow$$

$$u \cdot \frac{\partial z}{\partial v} + v \cdot \frac{\partial z}{\partial u} = u\left(2v f_y - 2v f_x\right) + v\left(2u f_x - 2u f_y\right)$$
$$= \cancel{2uv} f_y - \cancel{2uv} f_x + \cancel{2uv} f_x - \cancel{2uv} f_y = 0 .$$

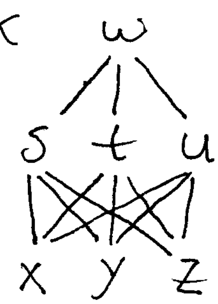
b.) Let $f(x,y) = \sin(x+2y) \rightarrow$

$$\frac{\partial z}{\partial v} = z_x \cdot \frac{\partial x}{\partial v} + z_y \cdot \frac{\partial y}{\partial v} = \cos(x+2y)\cdot(-2v) + 2\cos(x+2y)\cdot(2v)$$
$$= 2v \cos(x+2y) ,$$

$$\frac{\partial z}{\partial u} = z_x \cdot \frac{\partial x}{\partial u} + z_y \cdot \frac{\partial y}{\partial u} = \cos(x+2y)(2u) + 2\cos(x+2y)\cdot(-2u)$$
$$= -2u \cos(x+2y) \rightarrow$$

$$u \frac{\partial z}{\partial v} + v \frac{\partial z}{\partial u} = u\left(2v \cos(x+2y)\right) + v\left(-2u \cos(x+2y)\right)$$
$$= 2uv \cos(x+2y) - 2uv \cos(x+2y) = 0$$

14.) $w = f(s,t,u)$, $s = x-y$, $t = y-z$, $u = z-x$



a.) $$\frac{\partial w}{\partial x} = f_s \cdot \frac{\partial s}{\partial x} + f_t \cdot \frac{\partial t}{\partial x} + f_u \cdot \frac{\partial u}{\partial x}$$
$$= f_s(1) + f_t(0) + f_u(-1)$$
$$= f_s - f_u \quad ;$$

$$\frac{\partial w}{\partial y} = f_s \cdot \frac{\partial s}{\partial y} + f_t \cdot \frac{\partial t}{\partial y} + f_u \cdot \frac{\partial u}{\partial y}$$
$$= f_s \cdot (-1) + f_t \cdot (1) + f_u \cdot (0) = f_t - f_s \quad ;$$

$$\frac{\partial w}{\partial z} = f_s \cdot \frac{\partial s}{\partial z} + f_t \cdot \frac{\partial t}{\partial z} + f_u \cdot \frac{\partial u}{\partial z}$$
$$= f_s \cdot (0) + f_t \cdot (-1) + f_u \cdot (1) = f_u - f_t \quad ;$$

then $$\frac{\partial w}{\partial x} + \frac{\partial w}{\partial y} + \frac{\partial w}{\partial z} = (\cancel{f_s} - \cancel{f_u}) + (\cancel{f_t} - \cancel{f_s}) + (\cancel{f_u} - \cancel{f_t}) = 0 .$$

16.) Let $z = f(y+mx)$, i.e., $z = f(r)$ and $r = y+mx$, then

$$z_x = f'(r)\cdot \frac{\partial r}{\partial x} = f'(y+mx)\cdot m,$$

$$z_y = f'(r)\cdot \frac{\partial r}{\partial y} = f'(y+mx)\cdot (1),$$

$$z_{xx} = m\cdot \frac{\partial}{\partial x} f'(y+mx)$$

$$= m\cdot f''(y+mx)\cdot m = m^2 f''(y+mx),$$

$$z_{yy} = \frac{\partial}{\partial y} f'(y+mx) = f''(y+mx)\cdot (1),$$

$$z_{xy} = \frac{\partial}{\partial y}\cdot m f'(y+mx) = m f''(y+mx)\cdot(1);$$

assume $a z_{xx} + b z_{xy} + c z_{yy} = 0 \rightarrow$

$am^2 f''(y+mx) + bm f''(y+mx) + c f''(y+mx) = 0 \rightarrow$

$(am^2 + bm + c)\cdot f''(y+mx) = 0 \rightarrow$

$am^2 + bm + c = 0$ (if $f''(y+mx) \neq 0$).

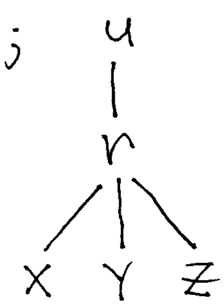
17.) a.) Let $z = f(x+y) \rightarrow z_x = f'(x+y)$,

$z_{xx} = f''(x+y)$, $z_{xy} = f''(x+y)$, $z_y = f'(x+y)$,

and $z_{yy} = f''(x+y)$, ~~then~~

$$z_{xx} - 2z_{xy} + z_{yy} = f''(x+y) - 2f''(x+y) + f''(x+y) = 0.$$

21.) Let $u = f(r)$ and $r = (x^2+y^2+z^2)^{1/2}$;



$$\frac{\partial u}{\partial x} = f'(r)\cdot \frac{\partial r}{\partial x} = f'(r)\cdot \frac{1}{2}(x^2+y^2+z^2)^{-1/2}(2x)$$

$$= f'(r)\cdot \frac{x}{\sqrt{x^2+y^2+z^2}};$$

$$\frac{\partial u}{\partial y} = f'(r)\cdot\frac{\partial r}{\partial y} = f'(r)\cdot\frac{1}{2}\left(x^2+y^2+z^2\right)^{-1/2}\cdot 2y$$

$$= f'(r)\cdot\frac{y}{\sqrt{x^2+y^2+z^2}} \quad ;$$

$$\frac{\partial u}{\partial z} = f'(r)\cdot\frac{\partial r}{\partial z} = f'(r)\cdot\frac{1}{2}\left(x^2+y^2+z^2\right)^{-1/2}\cdot 2z$$

$$= f'(r)\cdot\frac{z}{\sqrt{x^2+y^2+z^2}} \quad ;$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left[f'(r)\cdot\frac{x}{\sqrt{x^2+y^2+z^2}}\right]$$

$$= f'(r)\cdot\frac{\partial}{\partial x}\left(\frac{x}{\sqrt{x^2+y^2+z^2}}\right) + \frac{\partial}{\partial x}\left(f'(r)\right)\cdot\frac{x}{\sqrt{x^2+y^2+z^2}}$$

$$= f'(r)\cdot\frac{\sqrt{x^2+y^2+z^2}\cdot(1) - x\cdot\frac{1}{\cancel{2}}\left(x^2+y^2+z^2\right)^{-1/2}\cdot\cancel{2}x}{x^2+y^2+z^2}$$

$$+ f''(r)\cdot\frac{1}{\cancel{2}}\left(x^2+y^2+z^2\right)^{-1/2}(\cancel{2}x)\cdot\frac{x}{\sqrt{x^2+y^2+z^2}}$$

$$= f'(r)\cdot\frac{\dfrac{\sqrt{x^2+y^2+z^2}}{1} - \dfrac{x^2}{\sqrt{x^2+y^2+z^2}}}{x^2+y^2+z^2}$$

$$+ f''(r)\cdot\frac{x^2}{x^2+y^2+z^2}$$

$$= f'(r)\cdot\frac{y^2+z^2}{r^3} + f''(r)\cdot\frac{x^2}{r^2}$$; similarly,

$$\frac{\partial^2 u}{\partial y^2} = f'(r)\cdot\frac{x^2+z^2}{r^3} + f''(r)\cdot\frac{y^2}{r^2}$$ and

$$\frac{\partial^2 u}{\partial z^2} = f'(r)\cdot\frac{x^2+y^2}{r^3} + f''(r)\cdot\frac{z^2}{r^2}$$; thus,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \left\{ f'(r) \cdot \frac{y^2+z^2}{r^3} + f''(r) \cdot \frac{x^2}{r^2} \right\}$$

$$+ \left\{ f'(r) \cdot \frac{x^2+z^2}{r^3} + f''(r) \cdot \frac{y^2}{r^2} \right\}$$

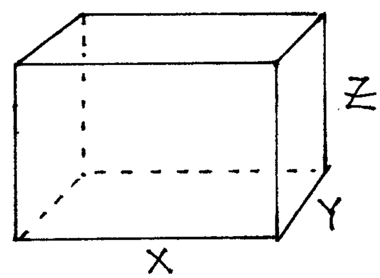
$$+ \left\{ f'(r) \cdot \frac{x^2+y^2}{r^3} + f''(r) \cdot \frac{z^2}{r^2} \right\}$$

$$= f'(r) \cdot \frac{2(x^2+y^2+z^2)}{r^3} + f''(r) \cdot \frac{x^2+y^2+z^2}{r^2}$$

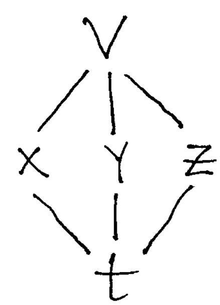
$$= f'(r) \cdot \frac{2r^2}{r^3} + f''(r) \cdot \cancel{\frac{r^2}{r^2}}$$

$$= \frac{du}{dr} \cdot \frac{2}{r} + \frac{d^2u}{dr^2} \quad .$$

22.) Volume $V = XYZ$; assume $\frac{dy}{dt} = 2$ ft./sec.



when $y = 3$ ft., $\frac{dx}{dt} = {}^-5$ ft./sec.
when $x = 8$ ft., $\frac{dz}{dt} = 2$ ft./sec. when $z = 4$ ft. ; find $\frac{dV}{dt}$:



$$\frac{dV}{dt} = V_X \frac{dx}{dt} + V_Y \cdot \frac{dY}{dt} + V_Z \cdot \frac{dz}{dt}$$

$$= YZ(^-5) + XZ(2) + XY(2)$$

$$= (3)(4)(^-5) + (8)(4)(2) + (8)(3)(2)$$

$$= {}^-60 + 64 + 48$$

$$= \mathbf{52} \text{ ft.}^3/\text{sec.}$$

23.) $T = f(x, y, z)$, $\frac{dx}{dt} = 4 \ \frac{mi.}{sec.}$, $\frac{dy}{dt} = 4 \ \frac{mi.}{sec.}$, $\frac{dz}{dt} = -3 \ \frac{mi.}{sec.}$, $T_x = 4$, $T_y = 7$, and $T_z = 9$;

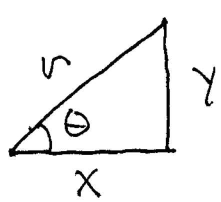
find $\frac{dT}{dt}$:

$$\frac{dT}{dt} = T_x \cdot \frac{dx}{dt} + T_y \cdot \frac{dy}{dt} + T_z \cdot \frac{dz}{dt}$$

$$= (4)(4) + (7)(4) + (9)(-3)$$

$$= 17 \text{ degrees/sec.}$$

26.)



$r = \sqrt{x^2+y^2}$,

$x = r\cos\theta$, $y = r\sin\theta$;

a.) $r = \sqrt{x^2+y^2} \longrightarrow \frac{\partial r}{\partial x} = \frac{1}{2}(x^2+y^2)^{-1/2}(2x)$

$$= \frac{x}{\sqrt{x^2+y^2}} = \frac{r\cos\theta}{r} = \cos\theta$$

b.) $r = \frac{x}{\cos\theta} \longrightarrow \frac{\partial r}{\partial x} = \frac{1}{\cos\theta}$

c.) Partial derivatives are dependent on the functions variables. In part a.) r is a function of x and y. In part b.) r is a function of x and θ.