Section 14.9 Solutions

2.) $z = x^2 - y^2 \rightarrow z_x = 2x = 0 \rightarrow \boxed{x=0}$,
$z_y = -2y = 0 \rightarrow \boxed{y=0}$ so critical point is $(0,0)$; $z_{xx} = 2$, $z_{yy} = -2$, $z_{xy} = 0$ then
$D = z_{xx} z_{yy} - (z_{xy})^2 = (2)(-2) - (0)^2 = -4 < 0$
so z has a saddle point at $(0,0)$, where $z = 0$.

4.) $z = x^4 + 8x^2 + y^2 - 4y \rightarrow z_x = 4x^3 + 16x = 0 \rightarrow$
$4x(x^2+4) = 0 \rightarrow \boxed{x=0}$, $z_y = 2y - 4 = 0 \rightarrow \boxed{y=2}$
so critical point is $(0,2)$; $z_{xx} = 12x^2 + 16$,
$z_{yy} = 2$, $z_{xy} = 0$ then
$D = z_{xx} z_{yy} - (z_{xy})^2 = (16)(2) - (0)^2 = 32 > 0$
and $z_{xx} = 16 > 0$ so z has a relative minimum at $(0,2)$, where $z = -4$.

9.) $z = \frac{4}{x} + \frac{2}{y} + xy \rightarrow z_x = \frac{-4}{x^2} + y = 0 \rightarrow$
$\boxed{y = \frac{4}{x^2}}$, $z_y = \frac{-2}{y^2} + x = 0 \rightarrow \boxed{x = \frac{2}{y^2}}$;

combine getting $y = \frac{4}{\left(\frac{2}{y^2}\right)^2} = \frac{4}{\frac{4}{y^4}} = y^4 \rightarrow$
$y^4 - y = 0 \rightarrow y(y^3-1) = 0 \rightarrow \boxed{y=0 \text{ or } y=1}$;
if $y=0$, this is impossible; if $\boxed{y=1}$ then
$\boxed{x=2}$ so critical point is $(2,1)$;
$z_{xx} = \frac{8}{x^3}$, $z_{yy} = \frac{4}{y^3}$, $z_{xy} = 1$ then
$D = z_{xx} z_{yy} - (z_{xy})^2 = (1)(4) - (1)^2 = 4 > 0$
and $z_{xx} = 1 > 0$ so z has a relative

minimum at (2,1), where $z = 6$.

10.) $z = x^3 - y^3 + 3xy \rightarrow z_x = 3x^2 + 3y = 0 \rightarrow$
$\boxed{y = -x^2}$, $z_y = -3y^2 + 3x = 0 \rightarrow \boxed{x = y^2}$;
combine getting $y = -(y^2)^2 = -y^4 \rightarrow$
$y^4 + y = 0 \rightarrow y(y^3+1) = 0 \rightarrow \boxed{y = 0 \text{ or } y = -1}$;
if $y=0$ then $x=0$; if $y=-1$ then $x=1$ so
critical points are (0,0) and (1,−1);

$z_{xx} = 6x$, $z_{yy} = -6y$, $z_{xy} = 3$;

For (0,0): $D = z_{xx} z_{yy} - (z_{xy})^2 = (0)(0) - (3)^2 = -9 < 0$
so z has a saddle point at (0,0),
where $z = 0$;
For (1,−1): $D = z_{xx} z_{yy} - (z_{xy})^2 = (6)(6) - (3)^2 = 27 > 0$
and $z_{xx} = 6 > 0$ so z has a relative
minimum at (1,−1), where $z = -1$.

12.) $z_{xx} z_{yy} - (z_{xy})^2 = (2)(4) - (-3)^2 = -1 < 0$
so z has a saddle point at (a,b).

14.) $z_{xx} z_{yy} - (z_{xy})^2 = (3)(4) - (2)^2 = 8 > 0$ and
$z_{xx} = 3 > 0$ so z has a relative minimum
at (a,b).

15.) $z_{xx} z_{yy} - (z_{xy})^2 = (-3)(-4) - (-2)^2 = 8 > 0$ and
$z_{xx} = -3 < 0$ so z has a relative maximum
at (a,b).

20.) $z = 6XY - X^2Y - XY^2 \rightarrow z_X = 6Y - 2XY - Y^2 = 0 \rightarrow$
$Y(6-2X-Y) = 0 \rightarrow \boxed{Y=0 \text{ or } Y = 6-2X}$,
$z_Y = 6X - X^2 - 2XY = 0 \rightarrow X(6-X-2Y) = 0 \rightarrow$
$\boxed{X=0 \text{ or } X = 6-2Y}$; there are four possibilities : $X=0$ and $Y=0$ so $\underline{(0,0)}$ is a critical point $\underline{\underline{\text{or}}}$ $X=0$ and $Y=6-2X \rightarrow$ $X=0$ and $Y=6$ so $\underline{(0,6)}$ is a critical point $\underline{\underline{\text{or}}}$ $X=6-2Y$ and $Y=0 \rightarrow X=6$ and $Y=0$ so $\underline{(6,0)}$ is a critical point $\underline{\underline{\text{or}}}$ $X=6-2Y$ and $Y=6-2X \rightarrow$
$Y = 6-2(6-2Y) = 6-12+4Y \rightarrow 3Y = 6 \rightarrow Y=2$ and $X=2$ so $\underline{(2,2)}$ is a critical point ;
$z_{XX} = -2Y$, $z_{YY} = -2X$, $z_{XY} = 6-2X-2Y$;

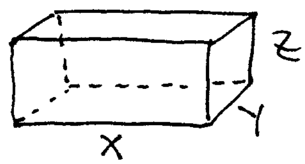
$\underline{\underline{\text{For } (0,0)}}$: $z_{XX}z_{YY} - (z_{XY})^2 = (0)(0) - (6)^2 = -36 < 0$ so z has a $\underline{\text{saddle point}}$ at $(0,0)$, where $z=0$;

$\underline{\underline{\text{For } (0,6)}}$: $z_{XX}z_{YY} - (z_{XY})^2 = (-12)(0) - (6)^2 = - \quad < 0$ so z has a $\underline{\text{saddle point}}$ at $(0,6)$, where $z=0$;

$\underline{\underline{\text{For } (6,0)}}$: $z_{XX}z_{YY} - (z_{XY})^2 = (0)(-12) - (-6)^2 = -36 < 0$ so z has a $\underline{\text{saddle point}}$ at $(6,0)$, where $z=0$;

$\underline{\underline{\text{For } (2,2)}}$: $z_{XX}z_{YY} - (z_{XY})^2 = (-4)(-4) - (-2)^2 = 12 > 0$ and $z_{XX} = -4 < 0$ so z has a $\underline{\text{relative maximum}}$ at $(2,2)$, where $z = 8$.

22.) $z = 2^{xy} \rightarrow z_x = 2^{xy} \cdot y \cdot \ln 2 = 0 \rightarrow \boxed{y=0}$,
$z_y = 2^{xy} \cdot x \cdot \ln 2 = 0 \rightarrow \boxed{x=0}$ so critical point is $\underline{(0,0)}$; $z_{xx} = 2^{xy} \cdot y^2 \cdot (\ln 2)^2$,
$z_{yy} = 2^{xy} \cdot x^2 \cdot (\ln 2)^2$, $z_{xy} = 2^{xy} \cdot \ln 2 + 2^{xy} \cdot xy \ln 2$;
then $D = z_{xx} z_{yy} - (z_{xy})^2$
$= (0)(0) - (\ln 2)^2 < 0$ so z has a
<u>saddle point</u> at $(0,0)$, where $z = 1$.

26.) 

Volume $xyz = 1 \rightarrow \boxed{z = 1/xy}$;
minimize cost

$C = C_{top} + C_{bottom} + C_{sides}$
$= 3(xy) + 3(xy) + 2(2xz + 2yz)$
$= 6xy + 4xz + 4yz$
$= 6xy + 4x(1/xy) + 4y(1/xy) \rightarrow$ cost

$\boxed{C = 6xy + \frac{4}{y} + \frac{4}{x}}$; find critical point $\rightarrow$

$C_x = 6y - \frac{4}{x^2} = 0 \rightarrow \boxed{y = \frac{2}{3x^2}}$ and

$C_y = 6x - \frac{4}{y^2} = 0 \rightarrow \boxed{x = \frac{2}{3y^2}} \rightarrow$

$x = \frac{2}{3\left(\frac{2}{3x^2}\right)^2} = \frac{2}{\frac{4}{3x^4}} = \frac{3}{2}x^4 \rightarrow 0 = \frac{3}{2}x^4 - x \rightarrow$

$0 = x\left(\frac{3}{2}x^3 - 1\right) \rightarrow x = 0$ (NO) or

$x^3 = \frac{2}{3} \rightarrow x = \left(\frac{2}{3}\right)^{1/3}$ ft. $\rightarrow$

$$y = \frac{2}{3\left(\left(\frac{2}{3}\right)^{1/3}\right)^2} = \frac{2}{3 \cdot \frac{2^{2/3}}{3^{2/3}}} = \frac{2^{1/3}}{3^{1/3}} \rightarrow Y = \left(\frac{2}{3}\right)^{1/3} ft. \rightarrow$$

$$z = \frac{1}{XY} = \frac{1}{\left(\frac{2}{3}\right)^{1/3}\left(\frac{2}{3}\right)^{1/3}} = \frac{1}{\left(\frac{2}{3}\right)^{2/3}} \rightarrow z = \left(\frac{3}{2}\right)^{2/3} ft. ,$$

and the minimum cost is

$$C = 6\left(\frac{2}{3}\right)^{1/3}\left(\frac{2}{3}\right)^{1/3} + \frac{4}{\left(\frac{2}{3}\right)^{1/3}} + \frac{4}{\left(\frac{2}{3}\right)^{1/3}}$$

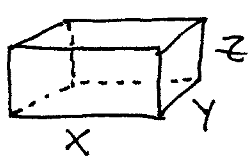
$$= (2)(3)\left(\frac{2}{3}\right)^{2/3} + 2^2 \cdot \frac{3^{1/3}}{2^{1/3}} + 2^2 \cdot \frac{3^{1/3}}{2^{1/3}}$$

$$= (2)(3) \cdot \frac{2^{2/3}}{3^{2/3}} + 2 \cdot 2^{5/3} \cdot 3^{1/3}$$

$$= 2^{5/3} \cdot 3^{1/3} + 2 \cdot 2^{5/3} \cdot 3^{1/3} = 3 \cdot 2^{5/3} \cdot 3^{1/3} \rightarrow$$

$$\boxed{C = 2^{5/3} 3^{4/3} ¢}$$

28.)



Surface area

$$2XY + 2XZ + 2YZ = 12 \ m^2$$

$$\rightarrow z(2X + 2Y) = 12 - 2XY \rightarrow$$

$$z = \frac{\cancel{2}(6 - XY)}{\cancel{2}(X+Y)} \rightarrow \boxed{z = \frac{6 - XY}{X+Y}} ;$$ maximize

volume $V = XYz = XY\left(\frac{6 - XY}{X+Y}\right) \rightarrow$

$$\boxed{V = \frac{6XY - X^2Y^2}{X+Y}} ;$$ find critical point $\rightarrow$

$$V_X = \frac{(X+Y)(6Y - 2XY^2) - (6XY - X^2Y^2)}{(X+Y)^2} = 0 \rightarrow$$

$$\cancel{6XY} + 6Y^2 - 2X^2Y^2 - 2XY^3 - \cancel{6XY} + X^2Y^2 = 0 \rightarrow$$

$$6Y^2 - X^2y^2 - 2Xy^3 = 0 \rightarrow Y^2(6 - X^2 - 2XY) = 0 \rightarrow$$

$$Y = 0 \text{ (NO)} \quad \text{or} \quad 6 - X^2 - 2XY = 0 \rightarrow$$

$$6 - X^2 = 2XY \rightarrow \boxed{y = \frac{6 - X^2}{2X}} \quad ;$$

$$V_Y = \frac{(X+Y)(6X - 2X^2Y) - (6XY - X^2Y^2)}{(X+Y)^2} = 0 \rightarrow$$

$$6X^2 - 2X^3y + \cancel{6XY} - 2X^2y^2 - \cancel{6XY} + X^2y^2 = 0 \rightarrow$$

$$6X^2 - 2X^3Y - X^2Y^2 = 0 \rightarrow X^2(6 - 2XY - Y^2) = 0 \rightarrow$$

$$X = 0 \text{ (NO)} \quad \text{or} \quad 6 - 2XY - Y^2 = 0 \rightarrow$$

$$6 - y^2 = 2XY \rightarrow \boxed{X = \frac{6 - y^2}{2y}} \quad ; \text{ combine}$$

equations $\rightarrow$

$$X = \frac{6 - y^2}{2Y} = \frac{6 - \left(\frac{6 - X^2}{2X}\right)^2}{2\left(\frac{6 - X^2}{2X}\right)} = \left[6 - \frac{(6 - X^2)^2}{(2X)^2}\right]\frac{X}{6 - X^2}$$

$$= \frac{6X}{6 - X^2} - \frac{\cancel{X}(6 - X^2)}{4X^{\cancel{2}}} = \frac{24X^2 - (X^4 - 12X^2 + 36)}{24X - 4X^3} \rightarrow$$

$$24X^2 - 4X^4 = -X^4 + 36X^2 - 36 \rightarrow$$

$$0 = 3X^4 + 12X^2 - 36 = 3(X^4 + 4X^2 - 12)$$

$$= 3(X^2 - 2)(X^2 + 6) \rightarrow X^2 = 2 \rightarrow X = -\sqrt{2} \text{ (NO)}$$

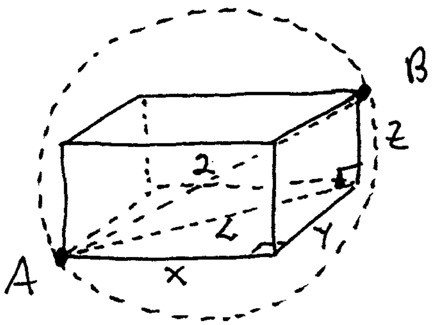
$$\text{or} \quad X = \sqrt{2} \text{ m.} \rightarrow y = \frac{6 - 2}{2\sqrt{2}} = \frac{2}{\sqrt{2}} \rightarrow$$

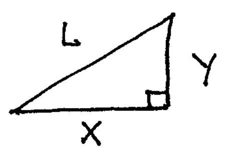
$$y = \sqrt{2} \text{ m.} \rightarrow Z = \frac{6 - \sqrt{2}\sqrt{2}}{\sqrt{2} + \sqrt{2}} = \frac{4}{2\sqrt{2}} \rightarrow$$

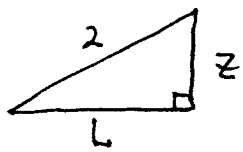
$Z = \sqrt{2}$ m. and maximum volume

$$V = 2\sqrt{2} \text{ m}^3 .$$

31.)







The distance from A to B is 2 ;

$x^2 + y^2 = L^2$ and

$L^2 + z^2 = 2^2 \rightarrow$

$x^2 + y^2 + z^2 = 4 \rightarrow$

$\boxed{z = \sqrt{4 - x^2 - y^2}}$;

maximize volume

$V = xyz = xy\sqrt{4-x^2-y^2} \rightarrow$

$V_x = xy \cdot \frac{1}{2}(4-x^2-y^2)^{-1/2}(-2x) + y\sqrt{4-x^2-y^2}$

$= \frac{-x^2y}{\sqrt{4-x^2-y^2}} + \frac{y\sqrt{4-x^2-y^2}}{1}$

$= \frac{-x^2y + y(4-x^2-y^2)}{\sqrt{4-x^2-y^2}}$

$= \frac{4y - 2x^2y - y^3}{\sqrt{4-x^2-y^2}} = 0 \rightarrow 4y - 2x^2y - y^3 = 0$

$\rightarrow y(4-2x^2-y^2) = 0 \rightarrow y = 0$ (NO) or

$4 - 2x^2 - y^2 = 0 \rightarrow 4 - 2x^2 = y^2 \rightarrow$

$\boxed{y = \sqrt{4-2x^2}}$; similarly,

$\boxed{x = \sqrt{4-2y^2}}$; combine equations $\rightarrow$

$x = \sqrt{4 - 2(4-2x^2)} = \sqrt{4x^2-4} \rightarrow x^2 = 4x^2 - 4 \rightarrow$

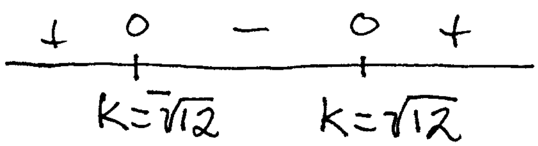
$3x^2 = 4 \rightarrow x^2 = \frac{4}{3} \rightarrow$ $x = \frac{2}{\sqrt{3}}$ $\rightarrow$

$y = \sqrt{4 - 2\left(\frac{4}{3}\right)} = \sqrt{\frac{4}{3}} \rightarrow$ $y = \frac{2}{\sqrt{3}}$ $\rightarrow$

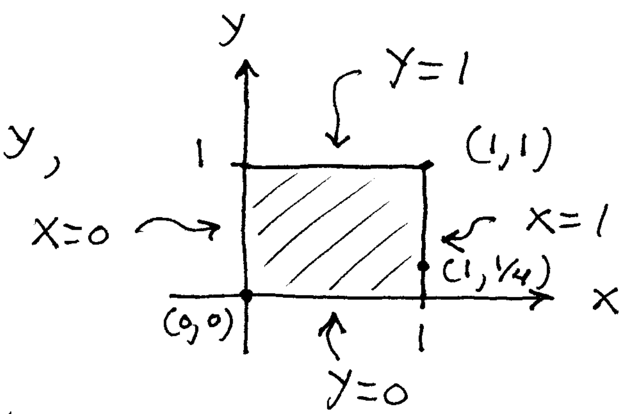
$z = \sqrt{4 - \frac{4}{3} - \frac{4}{3}} = \sqrt{\frac{4}{3}} \rightarrow$ $z = \frac{2}{\sqrt{3}}$ and

largest volume $V = \frac{8}{3\sqrt{3}}$.

32.) Let $z = x^2 + kxy + 3y^2$, find k so that z has a relative minimum at $(0,0)$:
$z_x = 2x + ky$, $z_y = kx + 6y$, $z_{xx} = 2$, $z_{yy} = 6$, $z_{xy} = k$ so relative minimum means $D = z_{xx} z_{yy} - (z_{xy})^2 > 0$
$\rightarrow (2)(6) - (k)^2 > 0 \rightarrow k^2 < 12 \rightarrow$
$k^2 - 12 < 0 \rightarrow (k - \sqrt{12})(k + \sqrt{12}) < 0$



$\rightarrow$ $-\sqrt{12} < k < +\sqrt{12}$.



36.) $z = 3x^2 - 4y^2 + 2xy$,
$z_x = 6x + 2y = 0 \rightarrow$
$\underline{y = -3x}$,
$z_y = -8y + 2x = 0 \rightarrow$
$\underline{x = 4y}$; combine equations $x = 4(-3x) = -12x$
$\rightarrow 13x = 0 \rightarrow x = 0, y = 0$ so $\boxed{(0,0)}$ is

a critical point; now look for critical points on the boundary of the square:

<u>along $y=0$</u>: $z = 3x^2 \rightarrow z' = 6x = 0 \rightarrow$ $x=0,\ y=0$ (already found);

<u>along $x=1$</u>: $z = 3 - 4y^2 + 2y \rightarrow z' = -8y+2=0$ $\rightarrow y = \frac{1}{4},\ x=1$ so $\boxed{(1, 1/4)}$ is a critical point;

<u>along $y=1$</u>: $z = 3x^2 - 4 + 2x \rightarrow z' = 6x+2 = 0$ $\rightarrow x = -1/3$ (NOT in domain);

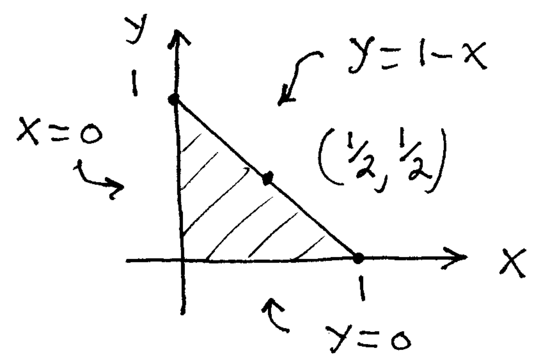
<u>along $x=0$</u>: $z = -4y^2 \rightarrow z' = -8y = 0 \rightarrow$ $y=0,\ x=0$ (already found);

	points	z-value	
corners	$(0,0)$	0	
	$(1,0)$	3	
	$(1,1)$	1	
	$(0,1)$	-4	absolute minimum
	$(1, 1/4)$	$13/4$	absolute maximum

37.) $z = xy$,

$z_x = y = 0$, $z_x = y = 0$

$\rightarrow$ <u>$(0,0)$</u> is critical point; now look for critical points on the boundary of the triangle:

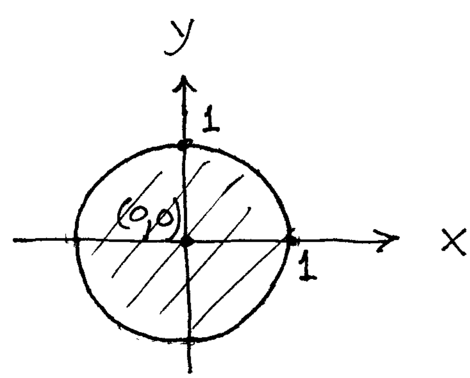


along $y=0$: $z=0$ for all $0 \le x \le 1$;
along $x=0$: $z=0$ for all $0 \le y \le 1$;
along $y=1-x$: $z = x(1-x) = x - x^2 \rightarrow$
$z' = 1-2x = 0 \rightarrow x = \frac{1}{2}, y = \frac{1}{2} \rightarrow$ critical point is $(\frac{1}{2}, \frac{1}{2})$;

points	z-value	
$(0,0)$	0	absolute minimum
$(\frac{1}{2}, \frac{1}{2})$	$\frac{1}{4}$	absolute maximum
$(x,0)$, $0 \le x \le 1$	0	absolute minimum
$(0,y)$, $0 \le y \le 1$	0	absolute minimum

39.) a.) $z = x^2 - y^2 + 2xy \rightarrow z_x = 2x + 2y = 0 \rightarrow \boxed{y = -x}$, $z_y = -2y + 2x = 0 \rightarrow \boxed{y = x} \rightarrow x = -x \rightarrow$
$2x = 0 \rightarrow x = 0, y = 0 \rightarrow$ critical point is $(0,0)$; $z_{xx} = 2$, $z_{yy} = -2$, $z_{xy} = 2 \rightarrow$
$D = z_{xx} z_{yy} - (z_{xy})^2 = (2)(-2) - (2)^2 = -8 < 0$
so z has a saddle point at $(0,0)$, where $z = 0$.

c.) Parameterize the circle $x^2 + y^2 = 1$:



$$\begin{cases} x = \cos t \\ y = \sin t \end{cases} \text{ for } 0 \le t < 2\pi,$$

$z = x^2 - y^2 + 2xy$
$= \cos^2 t - \sin^2 t + 2 \cos t \sin t$
$= \cos 2t + \sin 2t \rightarrow$

$z' = -2\sin 2t + 2\cos 2t = 0 \rightarrow$

$\cos 2t = \sin 2t \rightarrow 1 = \dfrac{\sin 2t}{\cos 2t} = \tan 2t$

$\rightarrow 2t = \dfrac{\pi}{4}, \dfrac{5\pi}{4}, \dfrac{9\pi}{4}, \dfrac{13\pi}{4}$

$\rightarrow t = \dfrac{\pi}{8}, \dfrac{5\pi}{8}, \dfrac{9\pi}{8}, \dfrac{13\pi}{8}$;

t	point	z-value	
----	$(0,0)$	0	
$\pi/8$	-----	$\sqrt{2}$	absolute maximum
$5\pi/8$	-----	$-\sqrt{2}$	absolute minimum
$9\pi/8$	-----	$\sqrt{2}$	absolute maximum
$13\pi/8$	-----	$-\sqrt{2}$	absolute minimum

48.) $w = xyz$ and $x+y+z=1$, where $x \ge 0, y \ge 0, z \ge 0 \rightarrow z = 1-x-y \rightarrow$
$w = xy(1-x-y) = xy - x^2y - xy^2 \rightarrow$
$\boxed{w = xy - x^2y - xy^2} \rightarrow$
$w_x = y - 2xy - y^2 = y(1-2x-y) = 0 \rightarrow$
$\boxed{y=0 \text{ or } y = 1-2x}$, $w_y = x - x^2 - 2xy$
$= x(1-x-2y) = 0 \rightarrow \boxed{x=0 \text{ or } x = 1-2y}$; there are four possibilities : $x=0$ and $y=0 \rightarrow$ $\underline{(0,0)}$ is a critical point ; $x=0$ and $y = 1-2(0) = 1 \rightarrow$ $\underline{(0,1)}$ is a critical point ; $y=0$ and $x = 1-2(0) = 1 \rightarrow$ $\underline{(1,0)}$ is a critical point ; $y = 1-2x$ and $x = 1-2y \rightarrow$
$x = 1-2y = 1-2(1-2x) = 1-2+4x \rightarrow 1 = 3x \rightarrow$
$x = 1/3$ and $y = 1/3 \rightarrow$ $\underline{(1/3, 1/3)}$ is a critical

point; $w_{xx} = -2y$, $w_{yy} = -2x$, $w_{xy} = 1-2x-2y$;

<u>For $(0,0)$</u>: $w_{xx}w_{yy} - (w_{xy})^2 = (0)(0) - (1)^2 = -1 < 0$ so w has a <u>saddle point</u> at $(0,0)$, where $w = 0$;

<u>For $(0,1)$</u>: $w_{xx}w_{yy} - (w_{xy})^2 = (-2)(0) - (-1)^2 = -1 < 0$ so w has a <u>saddle point</u> at $(0,1)$, where $w = 0$;

<u>For $(1,0)$</u>: $w_{xx}w_{yy} - (w_{xy})^2 = (0)(-2) - (-1)^2 = -1 < 0$ so w has a <u>saddle point</u> at $(1,0)$, where $w = 0$;

<u>For $(1/3, 1/3)$</u>: $w_{xx}w_{yy} - (w_{xy})^2 = \left(-\frac{2}{3}\right)\left(-\frac{2}{3}\right) - \left(-\frac{1}{3}\right)^2 = \frac{1}{3} > 0$ and $w_{xx} = -2/3 < 0$ so w has a <u>relative maximum</u> at $(1/3, 1/3)$, where $w = 1/27$.