Math 21C
Kouba
The Lagrange Form of the Remainder for the Taylor Series

Question: For what x-values is a function $y=f(x)$ equal to its Taylor Series centered at $x=a$?

$$f(x) = \underbrace{f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n}_{P_n(x;a)}$$
$$\underbrace{+ \frac{f^{(n+1)}(a)}{(n+1)!}(x-a)^{n+1} + \frac{f^{(n+2)}(a)}{(n+2)!}(x-a)^{n+2} + \cdots}_{R_n(x;a)}$$

$P_n(x;a)$: Taylor polynomial of degree n ;
$R_n(x;a)$: Taylor remainder (error) ;

$$f(x) = P_n(x;a) + R_n(x;a) \Rightarrow$$
$$\lim_{n\to\infty} f(x) = \lim_{n\to\infty} P_n(x;a) + \lim_{n\to\infty} R_n(x;a) \Rightarrow$$
$$f(x) = (\text{Taylor series}) + (0) \Rightarrow$$

Answer: It must be those x-values for which $\lim_{n\to\infty} R_n(x;a) = 0$.

<u>Fact</u>: (Lagrange Form of Taylor Remainder)

$$R_n(x;a) = \frac{f^{(n+1)}(c_n)\cdot(x-a)^{n+1}}{(n+1)!}, \quad \text{where}$$

c_n is a number between a and x.

<u>Example</u>: Show that e^x is equal to its Maclaurin series for all values of x:

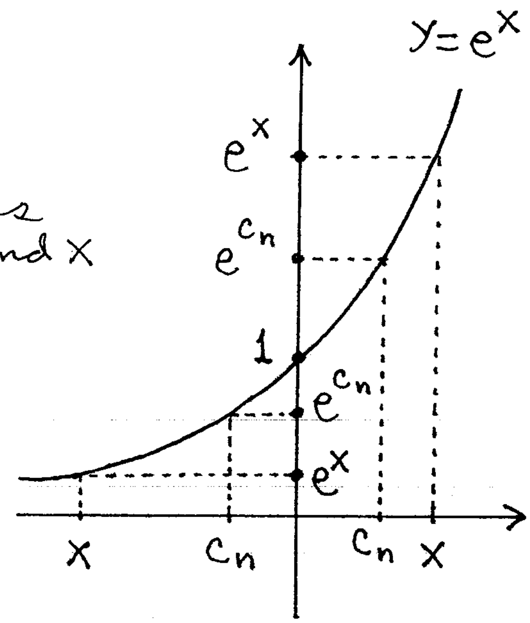
$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \quad ;$$

$f^{(n)}(x) = e^x$ for $n = 0, 1, 2, 3, \ldots$ then for any value of x

$$|R_n(x;0)| = \left|\frac{f^{(n+1)}(c_n)\cdot(x-0)^{n+1}}{(n+1)!}\right|$$

$$= e^{c_n}\cdot\frac{|x|^{n+1}}{(n+1)!}, \quad \text{where } c_n \text{ is between } 0 \text{ and } x$$

$$\le \begin{cases} 1\cdot\dfrac{|x|^{n+1}}{(n+1)!} & \text{if } x<0 \\[2ex] e^x\cdot\dfrac{|x|^{n+1}}{(n+1)!} & \text{if } x>0 \end{cases}.$$



Since $\lim\limits_{n\to\infty} \dfrac{|x|^{n+1}}{(n+1)!} = 0$ and

$\lim\limits_{n\to\infty} e^x\cdot\dfrac{|x|^{n+1}}{(n+1)!} = e^x\cdot(0) = 0$, it follows that

$\lim\limits_{n\to\infty} |R_n(x;0)| = 0$ so that $\lim\limits_{n\to\infty} R_n(x;0) = 0$.

Thus, $e^x = 1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \cdots$ for all values of x.