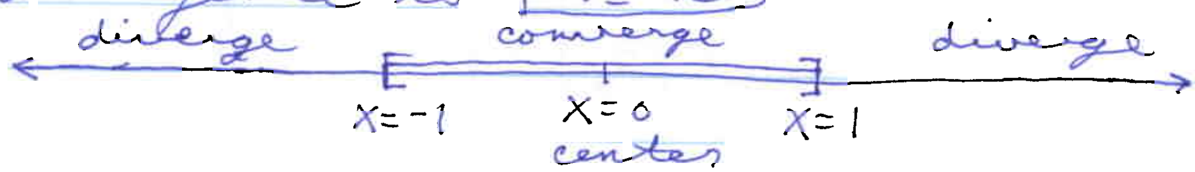
Section 11.4

1.) $\sum_{n=1}^{\infty} \frac{x^n}{n^2}$; $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{|x|^{n+1}}{(n+1)^2} \cdot \frac{n^2}{|x|^n}$

$= \lim_{n\to\infty} |x| \cdot \left(\frac{n}{n+1}\right)^2 = |x|\,(1)^2 = |x| < 1 \Rightarrow -1 < x < 1$;

if $x=1$: $\sum_{n=1}^{\infty} \frac{1^n}{n^2} = \sum_{n=1}^{\infty} \frac{1}{n^2}$ is a convergent p-series $(p = 2 > 1)$;

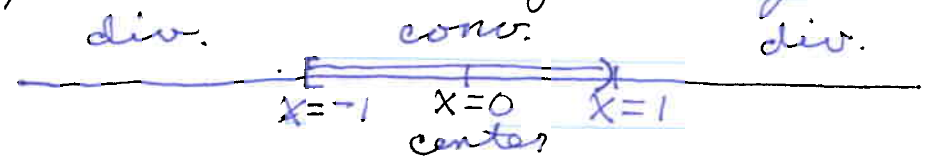
if $x=-1$: $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ converges by the alternating series test since $a_n = \frac{1}{n^2}$ is $+$, $\downarrow$, and $\lim_{n\to\infty} \frac{1}{n^2} = 0$; the interval of convergence is $\boxed{-1 \le x \le 1}$:



2.) $\sum_{n=1}^{\infty} \frac{x^n}{\sqrt{n}}$; $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{|x|^{n+1}}{\sqrt{n+1}} \cdot \frac{\sqrt{n}}{|x|^n}$

$= \lim_{n\to\infty} |x| \sqrt{\frac{n}{n+1}} = |x|\sqrt{1} = |x| < 1 \Rightarrow -1 < x < 1$;

if $x=1$: $\sum_{n=1}^{\infty} \frac{1^n}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges by p-series test $(p = \frac{1}{2} \le 1)$; if $x=-1$: $\lim_{n\to\infty} \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ converges by the alternating series test since $a_n = \frac{1}{\sqrt{n}}$ is $+$, $\downarrow$, and $\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$; the interval of convergence is $\boxed{-1 \le x < 1}$:



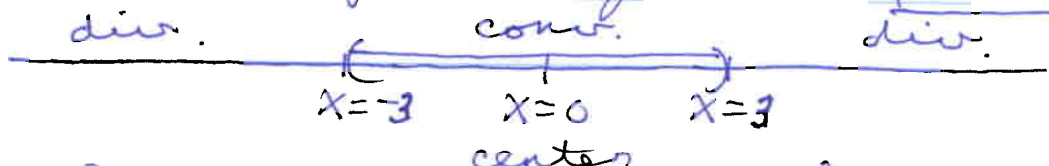
3.) $\sum_{n=0}^{\infty} \frac{x^n}{3^n}$; $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{|x|^{n+1}}{3^{n+1}} \cdot \frac{3^n}{|x|^n}$

$= \lim_{n\to\infty} \frac{|x|}{3} = \frac{|x|}{3} < 1 \Rightarrow |x| < 3 \Rightarrow -3 < x < 3$;

if $x = 3$: $\sum_{n=0}^{\infty} \frac{3^n}{3^n} = \sum_{n=0}^{\infty} 1$ diverges by the nth term test; if $x = -3$: $\sum_{n=0}^{\infty} \frac{(-3)^n}{3^n}$

$= \sum_{n=0}^{\infty} \left(\frac{-3}{3}\right)^n = \sum_{n=0}^{\infty} (-1)^n$ diverges by the nth term test since $\lim_{n\to\infty} (-1)^n$ DNE ($\neq 0$);

the interval of convergence is $\boxed{-3 < x < 3}$:

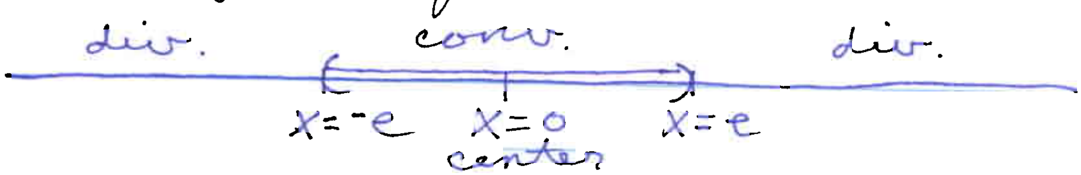


4.) $\sum_{n=1}^{\infty} \frac{n^2}{e^n} x^n$; $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{(n+1)^2}{e^{n+1}} |x|^{n+1} \cdot \frac{e^n}{n^2 |x|^n}$

$= \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^2 \cdot \frac{|x|}{e} = (1)^2 \frac{|x|}{e} = \frac{|x|}{e} < 1 \Rightarrow |x| < e \Rightarrow$

$-e < x < e$; if $x = e$: $\sum_{n=1}^{\infty} \frac{n^2}{e^n} \cdot e^n = \sum_{n=1}^{\infty} n^2$ diverges by the nth term test since $\lim_{n\to\infty} n^2 = \infty \neq 0$; if $x = -e$: $\sum_{n=1}^{\infty} \frac{n^2}{e^n} (-e)^n$

$= \sum_{n=1}^{\infty} n^2 (-1)^n$ diverges by the nth term test since $\lim_{n\to\infty} n^2 (-1)^n$ DNE ($\neq 0$); the interval of convergence is $\boxed{-e < x < e}$:



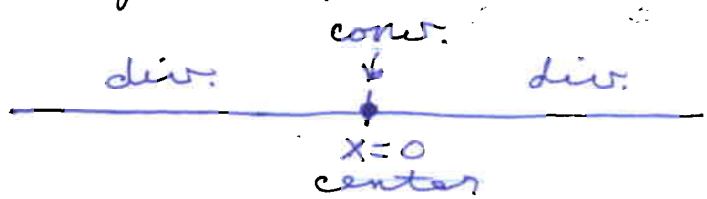
7.) $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{(2n)!}$; $\displaystyle\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{|x|^{n+1}}{(2(n+1))!} \cdot \frac{(2n)!}{|x|^n}$

$\displaystyle = \lim_{n\to\infty} |x| \cdot \frac{\cancel{(2n)(2n-1)(2n-2)\cdots 3\cdot 2\cdot 1}}{(2n+2)(2n+1)\cancel{(2n)(2n-1)\cdots 3\cdot 2\cdot 1}}$

$\displaystyle = \lim_{n\to\infty} \frac{|x|}{(2n+2)(2n+1)} = 0 < 1$ for all values of x.

10.) $\displaystyle\sum_{n=0}^{\infty} n!\,x^n$; $\displaystyle\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{(n+1)!\,|x|^{n+1}}{n!\,|x|}$

$\displaystyle = \lim_{n\to\infty} (n+1)|x| = \begin{cases} 0 & \text{if } x=0 \\ \infty & \text{if } x\neq 0 \end{cases}$, so series converges only for $\boxed{x=0}$;

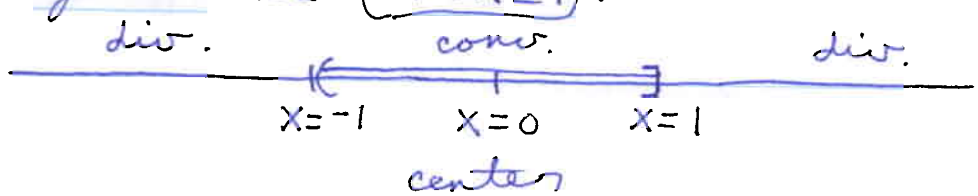


11.) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}x^n}{n}$; $\displaystyle\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{|x|^{n+1}}{n+1} \cdot \frac{n}{|x|^n}$

$\displaystyle = \lim_{n\to\infty} \frac{n}{n+1} \cdot |x| = (1)|x| = |x| < 1 \Rightarrow -1 < x < 1$;

if $x=1$: $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}(1)^n = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ converges by the alternating series test since $a_n = \frac{1}{n}$ is $+$, $\downarrow$, and $\displaystyle\lim_{n\to\infty} \frac{1}{n} = 0$; if $x=-1$: $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(-1)^n}{n}$

$\displaystyle = \sum_{n=1}^{\infty} \frac{(-1)^{2n+1}}{n} = \sum_{n=1}^{\infty} \frac{(-1)^{2n}(-1)}{n} = \sum_{n=1}^{\infty} \frac{-1}{n}$ diverges by the harmonic series test ; the interval of convergence is $\boxed{-1 < x \leq 1}$:



12.) $\sum_{n=1}^{\infty} \frac{2^n x^n}{n}$; $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{2^{n+1}|x|^{n+1}}{n+1} \cdot \frac{n}{2^n |x|^n}$

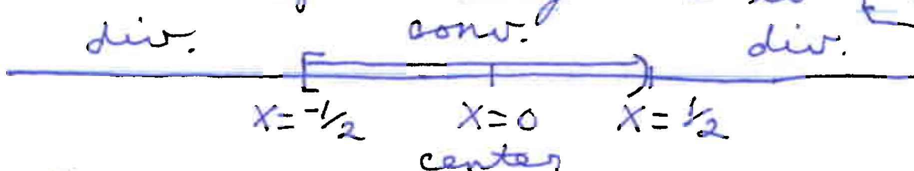
$= \lim_{n\to\infty} 2|x| \cdot \frac{n}{n+1} = 2|x|(1) = 2|x| < 1 \Rightarrow |x| < \frac{1}{2} \Rightarrow$

$-\frac{1}{2} < x < \frac{1}{2}$; if $x = \frac{1}{2}$: $\sum_{n=1}^{\infty} \frac{2^n (\frac{1}{2})^n}{n} = \sum_{n=1}^{\infty} \frac{(2\cdot\frac{1}{2})^n}{n}$

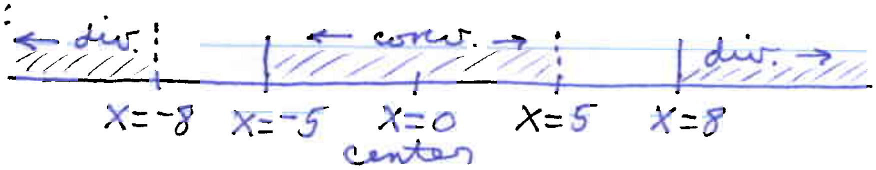
$= \sum_{n=1}^{\infty} \frac{1^n}{n} = \sum_{n=1}^{\infty} \frac{1}{n}$ diverges (harmonic series);

if $x = \frac{-1}{2}$: $\sum_{n=1}^{\infty} \frac{2^n (\frac{-1}{2})^n}{n} = \sum_{n=1}^{\infty} \frac{(2\cdot\frac{-1}{2})^n}{n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$

converges by the alternating series test since $a_n = \frac{1}{n}$ is $+$, $\downarrow$, and $\lim_{n\to\infty} \frac{1}{n} = 0$; the interval of convergence is $\boxed{-\frac{1}{2} \le x < \frac{1}{2}}$:



14.) $\sum_{n=0}^{\infty} a_n x^n$ (centered at $x=0$), converges at $x=-5$, diverges at $x=8$:



a.) $x=4$: convergence

b.) $x=4$: absolute convergence

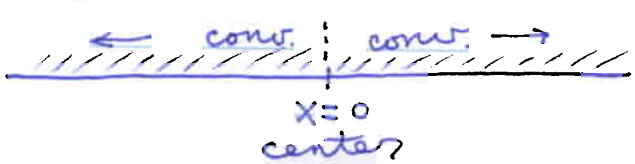
c.) $x=7$: unknown

d.) $x=-5$: absolute convergence unknown

e.) $x=-9$: diverges

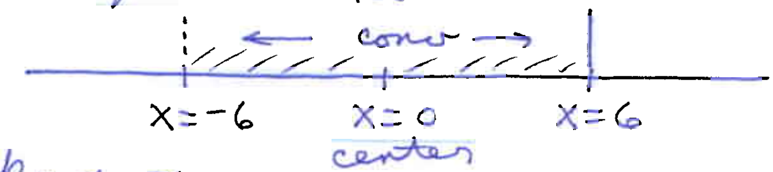
f.) $x=-8$: unknown

15.) $\sum_{n=0}^{\infty} a_n x^n$ (centered at $x=0$), converges for all $x>0$:



converges for all $x<0$.

16.) $\sum_{n=0}^{\infty} a_n 6^n$ converges ($\sum_{n=0}^{\infty} a_n x^n$ centered at $x=0$.):



a.) $x=-6$: unknown

b.) $x=5$: absolute convergence

c.) $x=-5$: absolute convergence

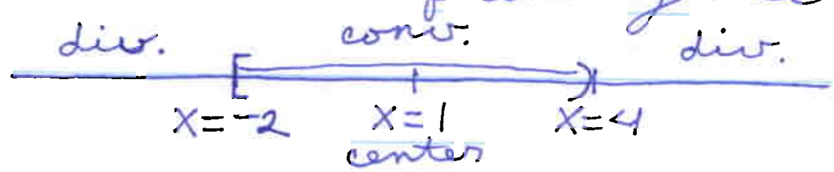
18.) $\sum_{n=1}^{\infty} \frac{(x-1)^n}{n\,3^n}$; $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{|x-1|^{n+1}}{(n+1)3^{n+1}} \cdot \frac{n\,3^n}{|x-1|^n}$

$= \lim_{n\to\infty} \frac{|x-1|}{3} \cdot \frac{n}{n+1} = \frac{|x-1|}{3}(1) = \frac{|x-1|}{3} < 1 \Rightarrow |x-1| < 3$

$\Rightarrow -3 < x-1 < 3 \Rightarrow -2 < x < 4$; if $x=4$:

$\sum_{n=1}^{\infty} \frac{3^n}{n\,3^n} = \sum_{n=1}^{\infty} \frac{1}{n}$ diverges (harmonic series);

if $x=-2$: $\sum_{n=1}^{\infty} \frac{(-3)^n}{n\,3^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges by the alternating series test since $a_n = \frac{1}{n}$ is $+, \downarrow$, and $\lim_{n\to\infty} \frac{1}{n} = 0$; the interval of convergence is $\boxed{-2 \le x < 4}$:



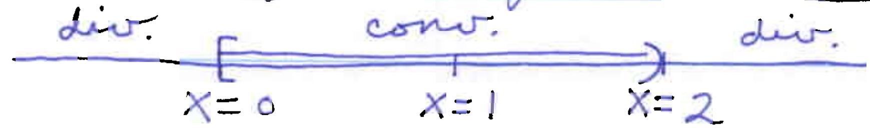
19.) $\sum_{n=0}^{\infty} \frac{(x-1)^n}{n+3}$; $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{|x-1|^{n+1}}{n+4} \cdot \frac{n+3}{|x-1|^n}$

$= \lim_{n\to\infty} |x-1| \cdot \frac{n+3}{n+4} = |x-1|(1) = |x-1| < 1 \Rightarrow -1 < x-1 < 1$

$\Rightarrow 0 < x < 2$; if $x=0$: $\sum_{n=0}^{\infty} \frac{(-1)^n}{n+3}$ converges by the alternating series test since $a_n = \frac{1}{n+3}$ is $+, \downarrow$, and $\lim_{n\to\infty} \frac{1}{n+3} = 0$; if $x=2$:

$\sum_{n=0}^{\infty} \frac{1^n}{n+3} = \sum_{n=0}^{\infty} \frac{1}{n+3}$ diverges (harmonic series);

the interval of convergence is $\boxed{0 \le x < 2}$:



22.) $\sum_{n=2}^{\infty} \frac{(x-5)^n}{n \ln n}$; $\lim_{n\to\infty} \frac{|x-5|^{n+1}}{(n+1)\ln(n+1)} \cdot \frac{n \ln n}{|x-5|^n}$

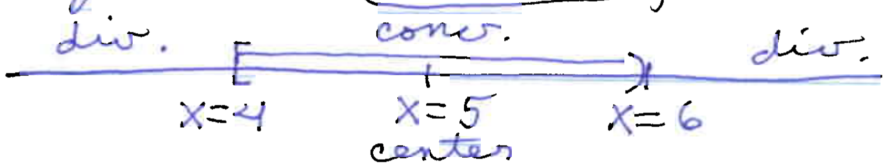
$= \lim_{n\to\infty} |x-5| \cdot \frac{n}{n+1} \cdot \frac{\ln n}{\ln(n+1)} = \lim_{n\to\infty} |x-5| \cdot (1) \cdot \frac{\frac{1}{n}}{\frac{1}{n+1}}$

$= \lim_{n\to\infty} |x-5| \cdot \frac{n+1}{n} = |x-5|\,(1) = |x-5| < 1 \Rightarrow$

$-1 < x-5 < 1 \Rightarrow 4 < x < 6$; if $x=4$: $\sum_{n=2}^{\infty} \frac{(-1)^n}{n \ln n}$ converges by the alternating series test since $a_n = \frac{1}{n \ln n}$ is $+, \downarrow$, and $\lim_{n\to\infty} \frac{1}{n \ln n} = 0$;

if $x=6$: $\sum_{n=2}^{\infty} \frac{1}{n \ln n} \rightarrow f(x) = \frac{1}{x \ln x}$ is $+, \downarrow$, and continuous and $\int_2^{\infty} \frac{1}{x \ln x}\,dx = \lim_{A\to\infty} \ln|\ln x|\Big|_2^A$

$= \lim_{A\to\infty} \left(\ln|\ln A| - \ln|\ln 2|\right) = \infty$, so $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ diverges by the integral test; the interval of convergence is $\boxed{4 \le x < 6}$;

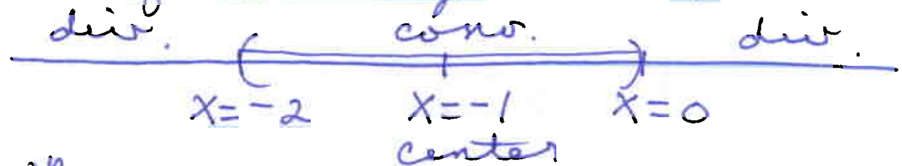


24.) $\sum_{n=1}^{\infty} n(x+1)^n$; $\lim_{n\to\infty} \frac{(n+1)|x+1|^{n+1}}{n\,|x+1|^n} = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)|x+1|$

$= (1)\,|x+1| < 1 \Rightarrow -1 < x+1 < 1 \Rightarrow -2 < x < 0$;

if $x=0$: $\sum_{n=1}^{\infty} n(1)^n = \sum_{n=1}^{\infty} n$ diverges by the nth term test since $\lim_{n\to\infty} n = \infty \neq 0$;

if $x=-2$: $\sum_{n=1}^{\infty} n(-1)^n$ diverges by the nth term test since $\lim_{n\to\infty} n(-1)^n$ DNE $(\neq 0)$; the interval of convergence is $\boxed{-2<x<0}$:



26.) $\sum_{n=0}^{\infty} (-1)^n \frac{(x+4)^n}{n+2}$; $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|}$

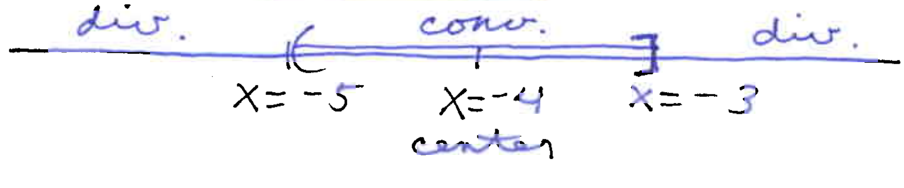
$= \lim_{n\to\infty} \frac{|x+4|^{n+1}}{n+3} \cdot \frac{n+2}{|x+4|^n} = \lim_{n\to\infty} \frac{1+\frac{2}{n}}{1+\frac{3}{n}} |x+4|$

$= (1)|x+4| = |x+4| < 1 \Rightarrow -1 < x+4 < 1 \Rightarrow -5 < x < -3$;

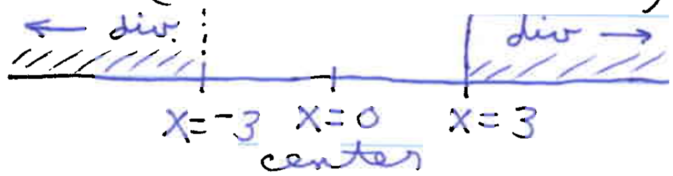
if $x=-3$: $\sum_{n=0}^{\infty} (-1)^n \frac{1^n}{n+2} = \sum_{n=0}^{\infty} (-1)^n \cdot \frac{1}{n+2}$ converges by the alternating series test since $a_n = \frac{1}{n+2}$ is $+$, $\downarrow$, and $\lim_{n\to\infty} \frac{1}{n+2} = 0$; if $x=-5$:

$\sum_{n=0}^{\infty} (-1)^n \frac{(-1)^n}{n+2} = \sum_{n=0}^{\infty} (-1)^{2n} \cdot \frac{1}{n+2} = \sum_{n=0}^{\infty} \left((-1)^2\right)^n \cdot \frac{1}{n+2}$

$= \sum_{n=0}^{\infty} 1^n \cdot \frac{1}{n+2} = \sum_{n=0}^{\infty} \frac{1}{n+2}$ verges by the harmonic series test ; the interval of convergence is $\boxed{-5<x\leq -3}$:

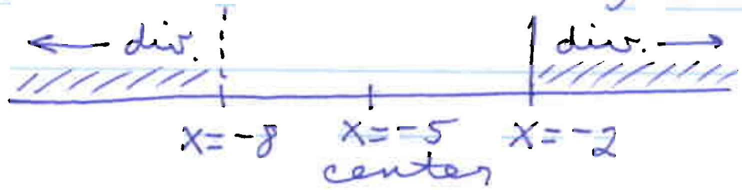


33.) a.) $\sum_{n=0}^{\infty} a_n x^5$ (centered at $x=0$) diverges at $x=3$:



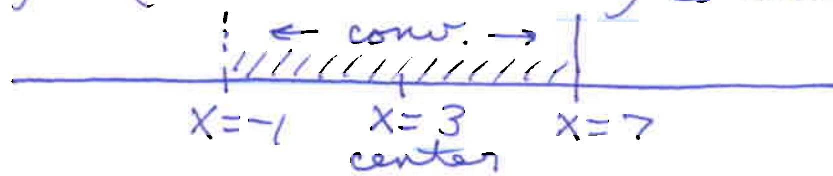
series must diverge at $x \geq 3$ and $x \leq -3$.

b.) $\sum_{n=0}^{\infty} a_n (x+5)^n$ (centered at $x=-5$) diverges at $x=-2$:



series must diverge for $x \geq -2$ and $x < -8$.

34.) $\sum_{n=0}^{\infty} a_n (x-3)^n$ (centered at $x=3$) converges at $x=7$:



series must converge for $-1 < x \leq 7$.