Section 11.1

Taylor series

(T) $f(x) = a_0 + a_1(x-a) + a_2(x-a)^2 + \cdots = \sum_{n=0}^{\infty} a_n (x-a)^n$

(S) $a_n = \dfrac{f^{(n)}(a)}{n!}$ for $n = 0, 1, 2, 3, \ldots$

(P) $P_n(x; a) = a_0 + a_1(x-a) + a_2(x-a)^2 + \cdots + a_n(x-a)^n$

Taylor polynomial

terminates at the nth power of $x-a$

13.) $f(x) = \ln(1+x)$,

$f'(x) = \dfrac{1}{1+x}$,

$f''(x) = \dfrac{-1}{(1+x)^2}$,

$f'''(x) = \dfrac{2}{(1+x)^3}$,

$f^{(4)}(x) = \dfrac{-3 \cdot 2}{(1+x)^4}$,

$\vdots$

$f^{(n)}(x) = \dfrac{(-1)^{n+1}(n-1)!}{(1+x)^n}$

for $n = 1, 2, 3, 4, \ldots$

$a_0 = \dfrac{f(0)}{0!} = \dfrac{\ln 1}{1} = \dfrac{0}{1} = 0$,

$a_1 = \dfrac{f'(0)}{1!} = \dfrac{1}{1} = 1$,

$a_2 = \dfrac{f''(0)}{2!} = \dfrac{-1}{2!} = \dfrac{-1}{2}$,

$a_3 = \dfrac{f'''(0)}{3!} = \dfrac{2}{3 \cdot 2} = \dfrac{1}{3}$,

$a_4 = \dfrac{f^{(4)}(0)}{4!} = \dfrac{-6}{24} = \dfrac{-1}{4}$,

$\vdots$

$a_n = \dfrac{f^{(n)}(0)}{n!} = \dfrac{(-1)^{n+1}(n-1)!}{n!} = \dfrac{(-1)^{n+1}}{n}$

$$\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n$$

15.) $f(x) = \cos x$, $f'(x) = -\sin x$, $f''(x) = -\cos x$, $f'''(x) = \sin x$, $f^{(4)}(x) = \cos x$, ... (repeat) ... ;

$a_0 = \dfrac{f(0)}{0!} = \dfrac{\cos 0}{1} = \dfrac{1}{1} = 1$, $a_1 = \dfrac{f'(0)}{1!} = \dfrac{-\sin 0}{1} = 0$,

$a_2 = \dfrac{f''(0)}{2!} = \dfrac{-\cos 0}{2!} = \dfrac{-1}{2!}$, $a_3 = \dfrac{f'''(0)}{3!} = \dfrac{\sin 0}{3!} = 0$,

$$a_5 = a_7 = a_9 = a_{11} = a_{odd} = 0 \quad ,$$

$$a_4 = \frac{f^{(4)}(0)}{4!} = \frac{\cos 0}{4!} = \frac{1}{4!}, \quad a_6 = \frac{-1}{6!}, \quad a_8 = \frac{1}{8!}, \cdots ;$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

17.) $f(x) = e^{-x}$, $f'(x) = -e^{-x}$, $f''(x) = e^{-x}$, $f'''(x) = -e^{-x}$, $\cdots$

$$a_0 = \frac{f(0)}{0!} = \frac{e^0}{1} = \frac{1}{1} = 1, \quad a_1 = \frac{f'(0)}{1!} = \frac{-e^0}{1} = -1,$$

$f^{(n)}(x) = (-1)^n e^{-x}$
$n = 0, 1, 2, \ldots$

$$a_2 = \frac{f''(0)}{2!} = \frac{e^0}{2!} = \frac{1}{2!}, \quad a_3 = \frac{f'''(0)}{3!} = \frac{-e^0}{3!} = \frac{-1}{3!}, \cdots$$

$$a_n = \frac{f^{(n)}(0)}{n!} = \frac{(-1)^n}{n!} \quad ; \text{ then}$$

$$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^n$$

18.) $f(x) = \frac{1}{1-x} = (1-x)^{-1}$,

$$f'(x) = -(1-x)^{-2}(-1) = (1-x)^{-2},$$

$$f''(x) = -2(1-x)^{-3}(-1) = 2(1-x)^{-3},$$

$$f'''(x) = -3\cdot 2(1-x)^{-4}(-1) = 3\cdot 2(1-x)^{-4},$$

$$\vdots$$

$$f^{(n)}(x) = n!\,(1-x)^{-(n+1)}$$

$$a_0 = \frac{f(0)}{0!} = \frac{1}{1} = 1,$$

$$a_1 = \frac{f'(0)}{1!} = \frac{1}{1} = 1,$$

$$a_2 = \frac{f''(0)}{2!} = \frac{2}{2} = 1,$$

$$a_3 = \frac{f'''(0)}{3!} = \frac{3\cdot 2}{6} = 1,$$

$$a_n = \frac{f^{(n)}(0)}{n!} = \frac{n!}{n!} = 1 ;$$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n .$$

2.) $f(x) = \frac{1}{1+x} = (1+x)^{-1}$, $f'(x) = -(1+x)^{-2}$, $f''(x) = 2(1+x)^{-3}$, $f'''(x) = -3\cdot 2(1+x)^{-4}$, $\vdots$ $f^{(n)}(x) = (-1)^n n!\,(1+x)^{-(n+1)}$

$a_0 = \frac{f(1)}{0!} = \frac{1}{2}$, $a_1 = \frac{f'(1)}{1!} = \frac{-1}{2^2}$, $a_2 = \frac{f''(1)}{2!} = \frac{2}{2^3\cdot 2} = \frac{1}{2^3}$, $a_3 = \frac{f'''(1)}{3!} = \frac{-3\cdot 2}{2^4\cdot 3!} = \frac{1}{2^4}, \ldots,$

$$a_n = \frac{f^{(n)}(1)}{n!} = (-1)^n\frac{n!\,2^{-(n+1)}}{n!} = \frac{(-1)^n}{2^{n+1}}$$; then

$$\frac{1}{1+x} = \frac{1}{2} - \frac{1}{2^2}(x-1) + \frac{1}{2^3}(x-1)^2 - \frac{1}{2^4}(x-1)^3 + \cdots$$
$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}}(x-1)^n$$; then

$P_1(x;1) = \frac{1}{2} - \frac{1}{2^2}(x-1)$, $P_2(x;1) = \frac{1}{2} - \frac{1}{2^2}(x-1) + \frac{1}{2^3}(x-1)^2$

3.) $P_1(x;0) = x$, $P_2(x;0) = x - \frac{1}{2}x^2$,
$P_3(x;0) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3$

6.) $f(x) = e^x$, $f'(x) = e^x$, $f^{(n)}(x) = e^x$ for $n = 0, 1, 2, 3, \ldots$,
$a_0 = \frac{f(2)}{0!} = \frac{e^2}{1} = e^2$, $a_1 = \frac{f'(2)}{1!} = \frac{e^2}{1} = e^2$,
$a_2 = \frac{f''(2)}{2!} = \frac{e^2}{2!}$, $a_3 = \frac{f'''(2)}{3!} = \frac{e^2}{3!}, \ldots$

$$a_n = \frac{f^{(n)}(2)}{n!} = \frac{e^2}{n!}$$; then

$$e^x = e^2 + e^2(x-2) + \frac{e^2}{2!}(x-2)^2 + \frac{e^2}{3!}(x-2)^3 + \cdots$$

$$= \sum_{n=0}^{\infty} \frac{e^2}{n!}(x-2)^n \quad ; \quad \text{then}$$

$$P_1(x;2) = e^2 + e^2(x-2), \quad P_2(x;2) = e^2 + e^2(x-2) + \frac{e^2}{2!}(x-2)^2,$$

$$P_3(x;2) = e^2 + e^2(x-2) + \frac{e^2}{2!}(x-2)^2 + \frac{e^2}{3!}(x-2)^3,$$

$$P_4(x;2) = e^2 + e^2(x-2) + \frac{e^2}{2!}(x-2)^2 + \frac{e^2}{3!}(x-2)^3 + \frac{e^2}{4}(x-2)^4$$

10.) $P_7(x;0) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}$

22.) The degree of $P_n(x;0)$ is $\leq n$.

27.) $f(x) = \sqrt{1+x}$,

$f'(x) = \frac{1}{2}(1+x)^{-1/2}$,

$f''(x) = \frac{-1}{2^2}(1+x)^{-3/2}$,

$f'''(x) = \frac{3}{2^3}(1+x)^{-5/2}$,

$f^{(4)}(x) = -\frac{3 \cdot 5}{2^4}(1+x)^{-7/2}$,

$\vdots$

$a_0 = \frac{f(0)}{0!} = \frac{1}{1} = 1$,

$a_1 = \frac{f'(0)}{1!} = \frac{1}{2}$,

$a_2 = \frac{f''(0)}{2!} = \frac{-1}{2^2 2!}$,

$a_3 = \frac{f'''(0)}{3!} = \frac{3}{2^3 3!}$,

$a_4 = \frac{f^{(4)}(0)}{4!} = \frac{-3 \cdot 5}{2^4 4!}$

$$f^{(n)}(x) = (-1)^{n+1} \frac{(1 \cdot 3 \cdot 5 \cdot 7 \cdots (2n-3))}{2^n}(1+x)^{-\frac{(2n-1)}{2}}$$

for $n = 2, 3, \ldots$ and

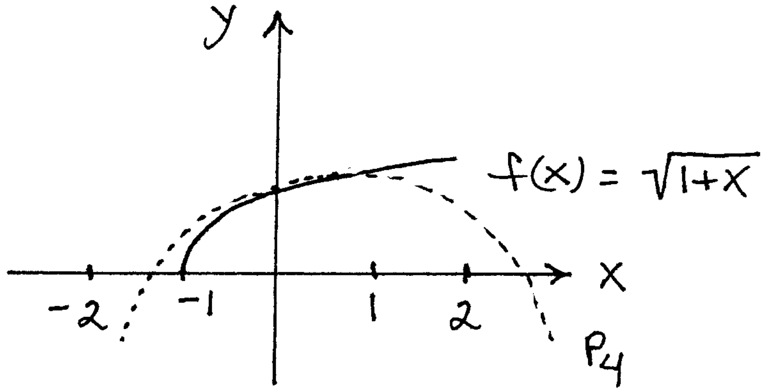
$$a_n = \frac{f^{(n)}(0)}{n!} = \frac{(-1)^{n+1}(1 \cdot 3 \cdot 5 \cdot 7 \cdots (2n-3))}{2^n \, n!},$$

$$\sqrt{1+x} = 1+\frac{1}{2}x - \frac{1}{2^2 2!}x^2 + \frac{3}{2^3 3!}x^3 - \cdots$$

$$= 1+\frac{1}{2}x + \sum_{n=2}^{\infty} (-1)^{n+1} \frac{1\cdot 3\cdot 5\cdot 7\cdots(2n-3)}{2^n n!} x^n$$

a.) $P_4(x;0) = 1+\frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \frac{5}{128}x^4$

b.)



34.) $f(x) = (1+x)^4$, $f'(x) = 4(1+x)^3$, $f''(x) = 12(1+x)^2$, $f'''(x) = 24(1+x)^1$, $f^{(4)}(x) = 24$, $f^{(5)}(x) = 0, \ldots;$

$a_0 = \frac{f(0)}{0!} = \frac{1}{1} = 1$, $a_1 = \frac{f'(0)}{1!} = \frac{4}{1} = 4$,

$a_2 = \frac{f''(0)}{2!} = \frac{12}{2} = 6$, $a_3 = \frac{f'''(0)}{3!} = \frac{24}{6} = 4$,

$a_4 = \frac{f^{(4)}(0)}{4!} = \frac{24}{24} = 1$, $a_5 = a_6 = a_7 = \cdots = 0$;

then

$$(1+x)^4 = 1+4x+6x^2+4x^3+x^4.$$