

Section 11.2

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \frac{f^{(n+1)}(a)}{(n+1)!}(x-a)^{n+1} + \dots$$

Taylor polynomial of degree n ,
 $P_n(x; a)$

Taylor remainder,
 $R_n(x; a)$

$$R_n(x; a) = \frac{f^{(n+1)}(c_n)}{(n+1)!} (x-a)^{n+1}$$

where constant

c_n is between a and x . Since
 $f(x) = P_n(x; a) + R_n(x; a)$, a function at x
 $f(x)$ and its Taylor series at x
 $\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n = \lim_{n \rightarrow \infty} P_n(x; a)$ are equal

at x iff $\lim_{n \rightarrow \infty} R_n(x; a) = 0$.

1.) $f(x) = \cos x$, $f'(x) = -\sin x$, $f''(x) = -\cos x$,
 $f'''(x) = \sin x$, $f^{(4)}(x) = \cos x$, ... then

$$|f^{(n)}(x)| = \begin{cases} |\pm \cos x| \leq 1 & \text{if } n=0, 2, 4, 6, \dots \\ |\pm \sin x| \leq 1 & \text{if } n=1, 3, 5, 7, \dots \end{cases};$$

$$|R_n(x; 0)| = \left| \frac{f^{(n+1)}(c_n)}{(n+1)!} (x-0)^{n+1} \right| = |f^{(n+1)}(c_n)| \cdot \frac{|x|^{n+1}}{(n+1)!}$$
$$\leq 1 \cdot \frac{|x|^{n+1}}{(n+1)!}; \text{ but } \lim_{n \rightarrow \infty} \frac{|x|^{n+1}}{(n+1)!} = 0 \text{ (by}$$

rule) for all x -values $\Rightarrow \lim_{n \rightarrow \infty} R_n(x; 0) = 0$
for all x -values. This means that

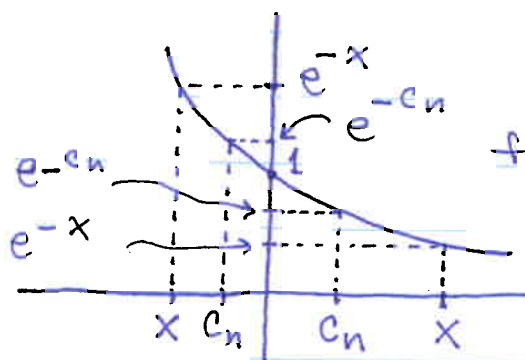
$f(x) = \cos x$ is equal to its Maclaurin series

$$1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots \text{ for all } x\text{-values.}$$

2.) $f(x) = e^{-x}$, $f'(x) = -e^{-x}$, $f''(x) = e^{-x}$, ...,
 $f^{(n)}(x) = (-1)^n e^{-x}$ for $n = 0, 1, 2, 3, \dots$; then

$$|R_n(x; 0)| = \left| \frac{f^{(n+1)}(c_n)}{(n+1)!} (x-0)^{n+1} \right| = \left| (-1)^n e^{-c_n} \right| \cdot \frac{|x|^{n+1}}{(n+1)!}$$

$$= e^{-c_n} \frac{|x|^{n+1}}{(n+1)!}, \text{ where } c_n \text{ is between } 0 \text{ and } x;$$



if $0 \leq c_n \leq x$,
 then $e^{-c_n} \leq 1$ and

$$e^{-c_n} \frac{|x|^{n+1}}{(n+1)!} \leq \frac{|x|^{n+1}}{(n+1)!}$$

where $\lim_{n \rightarrow \infty} \frac{|x|^{n+1}}{(n+1)!} = 0$; if $x \leq c_n \leq 0$,

then $e^{-c_n} \leq e^{-x}$ and $e^{-c_n} \frac{|x|^{n+1}}{(n+1)!} \leq e^{-x} \cdot \frac{|x|^{n+1}}{(n+1)!}$

where $\lim_{n \rightarrow \infty} e^{-x} \cdot \frac{|x|^{n+1}}{(n+1)!} = e^{-x} \cdot (0) = 0$.

For both cases, $\lim_{n \rightarrow \infty} e^{-c_n} \frac{|x|^{n+1}}{(n+1)!} = 0$, so

that $\lim_{n \rightarrow \infty} R_n(x; 0) = 0$. This means that

$f(x) = e^{-x}$ is equal to its Maclaurin series

$$1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \dots \text{ for all } x\text{-values}$$

$$7.) e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \Rightarrow$$

$$e^{-1} = \cancel{1} + \cancel{-1} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \frac{1}{6!} - \frac{1}{7!} + \dots ;$$

$$\frac{1}{6!} \approx 0.001 \text{ and } \frac{1}{7!} \approx 0.0002 \text{ so}$$

$$S_5 = \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \frac{1}{6!} \approx 0.368 \text{ estimates the}$$

value of e^{-1} with absolute error at most 0.0002 (to 3 decimal places).

$$\text{Problem A: } \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots \Rightarrow$$

$$\cos 1 = 1 - \frac{1}{2!} + \frac{1}{4!} - \frac{1}{6!} + \frac{1}{8!} - \frac{1}{10!} - \dots ;$$

$$\frac{1}{6!} \approx 0.001 \text{ and } \frac{1}{8!} \approx 0.00002 \text{ so}$$

$$S_4 = 1 - \frac{1}{2!} + \frac{1}{4!} - \frac{1}{6!} \approx 0.540 \text{ estimates the}$$

value of $\cos 1$ with absolute error at most 0.00002 (≤ 0.0001)

$$\text{Problem B: } f(x) = \ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$$

$$a.) \ln(2) = \ln(1+1) = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \dots ;$$

$$\frac{1}{n+1} \leq 0.001 \Rightarrow n+1 \geq 1000 \Rightarrow n \geq 999, \text{ so}$$

$S_{999} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots + \frac{1}{999}$ will estimate the value of $\ln(2)$ with absolute error at most 0.001.

$$b.) \ln(1.5) = \ln(1 + \frac{1}{2}) = \frac{1}{2} - \frac{1}{2 \cdot 2^2} + \frac{1}{3 \cdot 2^3} - \frac{1}{4 \cdot 2^4} + \dots ;$$

$$\frac{1}{7 \cdot 2^7} \approx 0.0011 \text{ and } \frac{1}{8 \cdot 2^8} \approx 0.0005 \text{ so}$$

$$S_7 = \frac{1}{2} - \frac{1}{2 \cdot 2^2} + \frac{1}{3 \cdot 2^3} - \frac{1}{4 \cdot 2^4} + \frac{1}{5 \cdot 2^5} - \frac{1}{6 \cdot 2^6} + \frac{1}{7 \cdot 2^7} \approx 0.406$$

estimates the value of $\ln(1.5)$ with absolute error at most 0.0005 (≤ 0.001).

$$8.) e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!} + \frac{x^{n+1}}{(n+1)!} + \frac{x^{n+2}}{(n+2)!} + \dots$$

$$= 1 + x + \frac{x^2}{2!} + \dots + \frac{x^n}{n!} + R_n(x; 0),$$

where $R_n(x; 0) = \frac{f^{(n+1)}(c_n)}{(n+1)!} (x-0)^{n+1}$, where

$0 \leq c_n \leq x$; let $x=2$ for e^2 then $0 \leq c_n \leq 2$

and
$$R_n(2; 0) = \frac{f^{(n+1)}(c_n)}{(n+1)!} 2^{n+1} = \frac{e^{c_n}}{(n+1)!} 2^{n+1}$$

$$\leq \frac{e^2}{(n+1)!} 2^{n+1} \leq \frac{(9) 2^{n+1}}{(n+1)!} \leq 0.0005?$$

| n | $\frac{(9) 2^{n+1}}{(n+1)!}$ |
|-----|------------------------------|
| 8 | 0.013 |
| 9 | 0.0025 |
| 10 | 0.00046 |

Choose $n=10$, then

$$S_{10} = 1 + 2 + \frac{2^2}{2!} + \frac{2^3}{3!} + \dots + \frac{2^{10}}{10!}$$

≈ 7.389 estimates the

value of e^2 with absolute

error at most 0.00046 (≤ 0.0005);

calculator: $e^2 = 7.38905$

$$14.) \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots \text{ then}$$

$$\int_0^1 \frac{1 - \cos x}{x} dx = \int_0^1 \frac{1 - \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots\right)}{x} dx$$

$$= \int_0^1 \frac{\frac{x^2}{2!} - \frac{x^4}{4!} + \frac{x^6}{6!} - \dots}{x} dx$$

$$= \int_0^1 \left(\frac{x}{2!} - \frac{x^3}{4!} + \frac{x^5}{6!} - \frac{x^7}{8!} + \dots \right) dx$$

$$= \frac{x^2}{2 \cdot 2!} - \frac{x^4}{4 \cdot 4!} + \frac{x^6}{6 \cdot 6!} - \frac{x^8}{8 \cdot 8!} + \dots \Big|_0^1$$

$$= \frac{1}{2 \cdot 2!} - \frac{1}{4 \cdot 4!} + \frac{1}{6 \cdot 6!} - \frac{1}{8 \cdot 8!} + \dots \text{ (an alt. series) ;}$$

$$\frac{1}{4 \cdot 4!} \approx 0.01 \text{ and } \frac{1}{6 \cdot 6!} \approx 0.0002, \text{ so}$$

$$S_2 = \frac{1}{2 \cdot 2!} - \frac{1}{4 \cdot 4!} \approx 0.2396 \text{ estimates the}$$

value of $\int_0^1 \frac{1 - \cos 2x}{x} dx$ with absolute error at most 0.0002 (to 3 decimal places)