Section 11.5

3.) a.) $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots$; Let $x = -t^2 \Rightarrow$

$$\frac{1}{1-(-t^2)} = 1 + (-t^2) + (-t^2)^2 + (-t^2)^3 + \cdots \Rightarrow$$

$$\frac{1}{1+t^2} = 1 - t^2 + t^4 - t^6 + t^8 - \cdots$$

b.) $$\int_0^x \frac{1}{1+t^2}\,dt = \int_0^x (1 - t^2 + t^4 - t^6 + \cdots)\,dt \Rightarrow$$

$$\arctan t\Big|_0^x = \left(t - \frac{t^3}{3} + \frac{t^5}{5} - \frac{t^7}{7} + \cdots\right)\Big|_0^x \Rightarrow$$

$$\arctan x - \overset{0}{\cancel{\arctan 0}} = \left(x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots\right) - (0) \Rightarrow$$

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$$

c.) $$\arctan x = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{2n+1}$$

d.) $$\arctan\left(\frac{1}{2}\right) = \frac{1}{2} - \frac{1}{2^3 \cdot 3} + \frac{1}{2^5 \cdot 5} - \frac{1}{2^7 \cdot 7} + \frac{1}{2^9 \cdot 9} - \frac{1}{2^{11} \cdot 11} + \cdots ;$$

$\frac{1}{2^7 \cdot 7} \approx 0.001$ and $\frac{1}{2^9 \cdot 9} \approx 0.0002$, so

$S_4 = \frac{1}{2} - \frac{1}{2^3 \cdot 3} + \frac{1}{2^5 \cdot 5} - \frac{1}{2^7 \cdot 7} \approx 0.4634$ estimates the exact value of $\arctan\left(\frac{1}{2}\right)$ with absolute error at most 0.0002 (≤ 0.0005);

calculator: $\arctan(1/2) \approx 0.4636$

6.) $$\frac{1-\cos x}{1-x^2} = \frac{1 - \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right)}{1-x^2}$$

$$= \frac{\frac{x^2}{2!} - \frac{x^4}{4!} + \frac{x^6}{6!} - \frac{x^8}{8!} + \cdots}{1-x^2}, \text{ so that}$$

$$\frac{x^2}{2} + \frac{11}{24}x^4 + \frac{331}{720}x^6 + \frac{3707}{8064}x^8 + \cdots$$

$$1-x^2 \;\Big)\; \frac{x^2}{2} - \frac{x^4}{24} + \frac{x^6}{720} - \frac{x^8}{40320} + \cdots$$

$$-\left(\frac{x^2}{2} - \frac{x^4}{2}\right)$$

$$\frac{11}{24}x^4 + \frac{x^6}{720}$$

$$-\left(\frac{11}{24}x^4 - \frac{11}{24}x^6\right)$$

$$\frac{331}{720}x^6 - \frac{x^8}{40320}$$

$$-\left(\frac{331}{720}x^6 - \frac{331}{720}x^8\right)$$

$$\frac{3707}{8064}x^8$$

and

$$\frac{1-\cos x}{1-x^2} = \frac{x^2}{2} + \frac{11}{24}x^4 + \frac{331}{720}x^6 + \frac{3707}{8064}x^8 + \cdots$$

7.) $e^x \sin x = \left(1 + x + \frac{x^2}{2} + \cdots\right)\left(x - \frac{x^3}{6} + \cdots\right)$

$= x + x^2 + \frac{x^3}{2} + \cdots + \frac{-x^3}{6} - \frac{x^4}{6} - \cdots$

$= x + x^2 + \frac{1}{3}x^3 + \cdots$

8.) $\frac{x}{\cos x} = \frac{x}{1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots}$, so that

$$x + \frac{x^3}{2} + \frac{5}{24}x^5 + \cdots$$

$$1 - \frac{x^2}{2} + \frac{x^4}{24} - \cdots \;\Big)\; x$$

$$-\left(x - \frac{x^3}{2} + \frac{x^5}{24} - \cdots\right)$$

$$\frac{x^3}{2} - \frac{x^5}{24} + \cdots$$

$$-\left(\frac{x^3}{2} - \frac{x^5}{4} + \cdots\right)$$

$$\frac{5}{24}x^5 -$$

, and

$$\frac{x}{\cos x} = x + \frac{x^3}{2} + \frac{5}{24}x^5 + \cdots$$

9.) $$\lim_{x\to 0} \frac{1-\cos x}{x^2} = \lim_{x\to 0} \frac{1-\left(1-\frac{x^2}{2!}+\frac{x^4}{4!}-\cdots\right)}{x^2}$$

$$= \lim_{x\to 0} \frac{\frac{x^2}{2!}-\frac{x^4}{4!}+\cdots}{x^2} = \lim_{x\to 0} \frac{\cancel{x^2}\left(\frac{1}{2}-\frac{x^2}{4!}+\frac{x^4}{6!}-\cdots\right)}{\cancel{x^2}}$$

$$= \frac{1}{2} .$$

11.) $$\lim_{x\to 0} \frac{\sin^2 x^3}{(1-\cos x^2)^3} = \lim_{x\to 0} \frac{\left((x^3)-\frac{(x^3)^3}{3!}+\frac{(x^3)^5}{5!}-\cdots\right)^2}{\left(1-\left(1-\frac{(x^2)^2}{2!}+\frac{(x^2)^4}{4!}-\cdots\right)\right)^3}$$

$$= \lim_{x\to 0} \frac{\left(x^3-\frac{x^9}{6}+\frac{x^{15}}{120}-\cdots\right)^2}{\left(\frac{x^4}{2}-\frac{x^8}{24}+\cdots\right)^3}$$

$$= \lim_{x\to 0} \frac{(x^3)^2\left(1-\frac{x^6}{6}+\frac{x^{12}}{120}-\cdots\right)^2}{(x^4)^3\left(\frac{1}{2}-\frac{x^4}{24}+\cdots\right)^3}$$

$$= \lim_{x\to 0} \frac{\left(1-\frac{x^6}{6}+\frac{x^{12}}{120}-\cdots\right)^2}{x^6\left(\frac{1}{2}-\frac{x^4}{24}+\cdots\right)^3} = \frac{1}{0^+} = \infty .$$

14.) $$\lim_{x\to 0} \frac{\sin x\,(1-\cos x)}{e^{x^3}-1}$$

$$= \lim_{x\to 0} \frac{\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}-\cdots\right)\left(1-\left(1-\frac{x^2}{2!}+\frac{x^4}{4!}-\cdots\right)\right)}{\left(1+(x^3)+\frac{(x^3)^2}{2!}+\frac{(x^3)^3}{3!}+\cdots\right)-1}$$

$$= \lim_{x\to 0} \frac{x\left(1-\frac{x^2}{3!}+\frac{x^4}{5!}-\cdots\right)\left(\frac{x^2}{2!}-\frac{x^4}{4!}+\cdots\right)}{x^3+\frac{x^6}{2!}+\frac{x^9}{3!}+\cdots}$$

$$= \lim_{x\to 0} \frac{\cancel{x}\left(1-\frac{x^2}{3!}+\frac{x^4}{5!}-\cdots\right)\cdot\cancel{x^2}\left(\frac{1}{2}-\frac{x^2}{4!}+\cdots\right)}{\cancel{x^3}\left(1+\frac{x^3}{2!}+\frac{x^6}{3!}+\cdots\right)} = \frac{1}{2}$$

15.) a.) $\int_0^1 \sqrt{x}\sin x\,dx = \int_0^1 \sqrt{x}\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right)dx$

$= \int_0^1 \left(x^{3/2} - \frac{x^{7/2}}{3!} + \frac{x^{11/2}}{5!} - \frac{x^{15/2}}{7!} + \cdots\right)dx$

$= \left(\frac{2}{5}x^{5/2} - \frac{2}{9}\frac{x^{9/2}}{3!} + \frac{2}{13}\frac{x^{13/2}}{5!} - \frac{2}{17}\frac{x^{17/2}}{7!} + \cdots\right)\Big|_0^1$

$= \left(\frac{2}{5} - \frac{2}{9\cdot 3!} + \frac{2}{13\cdot 5!} - \frac{2}{17\cdot 7!} + \cdots\right) - (0)$, and

b.) $\frac{2}{17\cdot 7!} \approx 0.00002$ so that

$S_3 = \frac{2}{5} - \frac{2}{9\cdot 3!} + \frac{2}{13\cdot 5!} \approx 0.36424$ estimates the exact value of $\int_0^1 \sqrt{x}\sin x\,dx$ with absolute error at most $0.00002\ (\le 0.00005)$;

<u>calculator</u>: $\int_0^1 \sqrt{x}\sin x\,dx \approx 0.36422$.

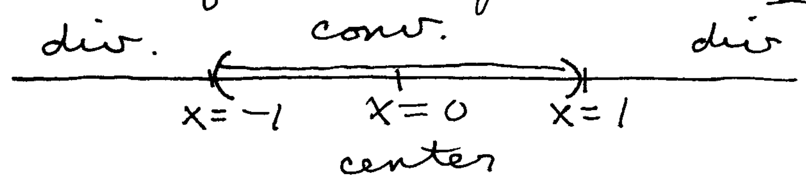
25.) a.) $\sum_{n=0}^{\infty} n^2 x^n$; $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{(n+1)^2 |x|^{n+1}}{n^2 |x|^n}$

$= \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^2 |x| = 1\cdot|x| = |x| < 1 \Rightarrow -1 < x < 1$;

if $x=1$: $\sum_{n=0}^{\infty} n^2(1)^n = \sum_{n=0}^{\infty} n^2$ diverges by the nth term test since $\lim_{n\to\infty} n^2 = \infty \neq 0$;

if $x=-1$: $\sum_{n=0}^{\infty} n^2(-1)^n$ diverges by the nth term test since $\lim_{n\to\infty} n^2(-1)^n$ DNE $(\neq 0)$;

the interval of convergence is $\boxed{-1 < x < 1}$:



b.) Let $f(x) = x^2 \cdot \frac{1}{1-x} = x^2(1+x+x^2+x^3+\cdots)$ for $|x|<1$

$$= x^2+x^3+x^4+x^5+\cdots \Rightarrow$$

$$f'(x) = \frac{(1-x)(2x)-x^2(-1)}{(1-x)^2} = \frac{2x-2x^2+x^2}{(1-x)^2} = \frac{2x-x^2}{(1-x)^2} \Rightarrow$$

$$\frac{2x-x^2}{(1-x)^2} = 2x+3x^2+4x^3+5x^4+\cdots \Rightarrow$$

$$x \cdot \frac{2x-x^2}{(1-x)^2} = \frac{2x^2-x^3}{(1-x)^2} = 2x^2+3x^3+4x^4+5x^5+\cdots \Rightarrow$$

$$D\,\frac{2x^2-x^3}{(1-x)^2} = \frac{(1-x)^2(4x-3x^2)-(2x^2-x^3)\cdot 2(1-x)(-1)}{(1-x)^4}$$

$$= \frac{x(1-x)\left[(1-x)(4-3x)+2(2x-x^2)\right]}{(1-x)^4}$$

$$= \frac{x\left[4-\cancel{4x}-3x+3x^2+\cancel{4x}-2x^2\right]}{(1-x)^3}$$

$$= \frac{x(x^2-3x+4)}{(1-x)^3} = \frac{x^3-3x^2+4x}{(1-x)^3} \Rightarrow$$

$$\frac{x^3-3x^2+4x}{(1-x)^3} = 2^2x+3^2x^2+4^2x^3+5^2x^4+\cdots \Rightarrow$$

$$\frac{x^4-3x^3+4x^2}{(1-x)^3} = 2^2x^2+3^2x^3+4^2x^4+5^2x^5+\cdots$$

c.) The formula will give the correct answer for $x=1/3$ since the interval of convergence is $-1<x<1$.

Chapter 11 Review Exercises

18.) $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$ so that

$e^{1/10} \approx 1 + \frac{1}{10} + \frac{1}{10^2 2!} = 1.105$; the error

$R_3(\frac{1}{10}; 0) = \frac{f^{(4)}(c_n)\,(\frac{1}{10} - 0)^4}{4!}$ where $0 \le c_n \le \frac{1}{10}$

$= \frac{e^{c_n}}{10^4 4!} < \frac{2}{10^4 4!} \approx 0.000008$

19.) $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$ so that

$\cos(\frac{1}{3}) \approx 1 - \frac{1}{3^2 2!} + \frac{1}{3^4 4!} \approx 0.945$; the absolute

error $|R_3| < \frac{1}{3^6 6!} \approx 0.000002$

33.) a.) $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \Rightarrow$

$\cos x^3 = 1 - \frac{(x^3)^2}{2!} + \frac{(x^3)^4}{4!} - \frac{(x^3)^6}{6!} + \cdots$

$= 1 - \frac{x^6}{2!} + \frac{x^{12}}{4!} - \frac{x^{18}}{6!} + \cdots \Rightarrow$

$\int_0^{\frac{1}{2}} \cos x^3 \, dx = \int_0^{1/2} (1 - \frac{x^6}{2!} + \frac{x^{12}}{4!} - \frac{x^{18}}{6!} + \cdots) \, dx$

$= (x - \frac{x^7}{7 \cdot 2!} + \frac{x^{13}}{13 \cdot 4!} - \frac{x^{19}}{19 \cdot 6!} + \cdots) \Big|_0^{1/2}$

$\approx \frac{1}{2} - \frac{1}{2^7 \cdot 7 \cdot 2!} + \frac{1}{2^{13} 13 \cdot 4!} \approx 0.4994$; the absolute

error $|R_3| < \frac{1}{2^{19} \cdot 19 \cdot 6!} \approx 0.0000000001$

d.) $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$ so that

$e^{-x^3} = 1 + (-x^3) + \frac{(-x^3)^2}{2!} + \frac{(-x^3)^3}{3!} + \cdots$

$$= 1 - x^3 + \frac{x^6}{2!} - \frac{x^9}{3!} + \cdots \Rightarrow$$

$$\int_1^2 e^{-x^3}\,dx = \int_1^2 \left(1 - x^3 + \frac{x^6}{2!} - \frac{x^9}{3!} + \cdots\right) dx$$

$$= \left(x - \frac{x^4}{4} + \frac{x^7}{7\cdot 2!} - \frac{x^{10}}{10\cdot 3!} + \cdots\right)\Big|_1^2$$

$$= \left(2 - \frac{2^4}{4} + \frac{2^7}{7\cdot 2!} - \frac{2^{10}}{10\cdot 3!} + \cdots\right) - \left(1 - \frac{1}{4} + \frac{1}{7\cdot 2!} - \frac{1}{10\cdot 3!} + \cdots\right)$$

$$\approx \left(2 - \frac{2^4}{4} + \frac{2^7}{7\cdot 2!}\right) - \left(1 - \frac{1}{4} + \frac{1}{7\cdot 2!}\right) \approx 6.321$$; the

errors for the 2 alternating series are $|R_3| < \frac{2^{10}}{10\cdot 3!} \approx 17.1$ and $|R_3| < \frac{1}{10\cdot 3!} \approx 0.017$ so that the total absolute error is

$$|R_3| \le 17.1 + 0.017 = 17.117$$

<u>calculator</u> : $\int_1^2 e^{-x^3}\,dx \approx 0.085$!

The estimate is a poor one because the interval $[1, 2]$ is "far" from the center $(x=0)$ of the Maclaurin series.

49.) $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots \Rightarrow$

$\cos x^2 = 1 - \frac{x^4}{2!} + \frac{x^8}{4!} - \cdots$; then

$$\lim_{x\to 0} \frac{(1-\cos x^2)^5}{(x - \sin x)^{20}} = \lim_{x\to 0} \frac{\left(\frac{x^4}{2!} - \frac{x^8}{4!} + \frac{x^{12}}{6!} - \cdots\right)^5}{\left(x - \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right)\right)^{20}}$$

$$= \lim_{x\to 0} \frac{\left(x^4\left(\frac{1}{2!} - \frac{x^4}{4!} + \frac{x^8}{6!} - \cdots\right)\right)^5}{\left(x^3\left(\frac{1}{3!} - \frac{x^2}{5!} + \frac{x^4}{7!} - \cdots\right)\right)^{20}}$$

$$= \lim_{x\to 0} \frac{x^{20}\left(\frac{1}{2!} - \frac{x^4}{4!} + \frac{x^8}{6!} - \cdots\right)^5}{x^{60}\left(\frac{1}{3!} - \frac{x^2}{5!} + \frac{x^4}{7!} - \cdots\right)^{20}}$$

$$= \lim_{x\to 0} \frac{1}{x^{40}} \cdot \frac{\left(\frac{1}{2!} - \frac{x^4}{4!} + \frac{x^8}{6!} - \cdots\right)^5}{\left(\frac{1}{3!} - \frac{x^2}{5!} + \frac{x^4}{7!} - \cdots\right)^{20}}$$

$$= \frac{1}{0^+} \cdot \frac{\left(\frac{1}{2}\right)^5}{\left(\frac{1}{6}\right)^{20}} = +\infty .$$

51.) $\sum_{n=0}^{\infty} 2^n x^n = \sum_{n=0}^{\infty} (2x)^n = 1 + (2x) + (2x)^2 + (2x)^3 + \cdots$

$= \frac{1}{1-(2x)} \Rightarrow f(x) = \frac{1}{1-2x}$; then

$f'(x) = D(1-2x)^{-1} = -1\cdot(1-2x)^{-2}(-2) = 2(1-2x)^{-2}$,

$f''(x) = -2^2(1-2x)^{-3}(-2) = 2^2 \cdot 2(1-2x)^{-3}$,

$f'''(x) = -2^2\cdot 3\cdot 2(1-2x)^{-4}\cdot(-2) = 2^3\, 3!\,(1-2x)^{-4}$,

$f^{(4)}(x) = -2^3\, 4!\,(1-2x)^{-5}(-2) = 2^4\, 4!\,(1-2x)^{-5}, \ldots,$

$f^{(n)}(x) = 2^n\, n!\,(1-2x)^{-(n+1)}$ so that

$f^{(33)}(0) = 2^{33}\, 33!\,(1)^{-34} = 2^{33} \cdot 33!$