Math 21C
Kouba
Moments and Center of Mass of a Flat Region R with Variable Density

Consider a flat region R in two-dimensional space with variable density (measured in mass/area units) $\delta(P)$ at the point $P = (x, y)$ in R. We seek to find the center of mass $(\bar{x}, \bar{y})$ of R. First partition R into n parts R_1, R_2, R_3, and R_n of areas A_1, A_2, A_3, and A_n, resp. Pick sampling points $P_i = (x_i, y_i)$ in region R_i for $i = 1, 2, 3, \cdots, n$. Consider the vertical line $x = \bar{x}$. Define the *moment* of region R_i about line $x = \bar{x}$ to be

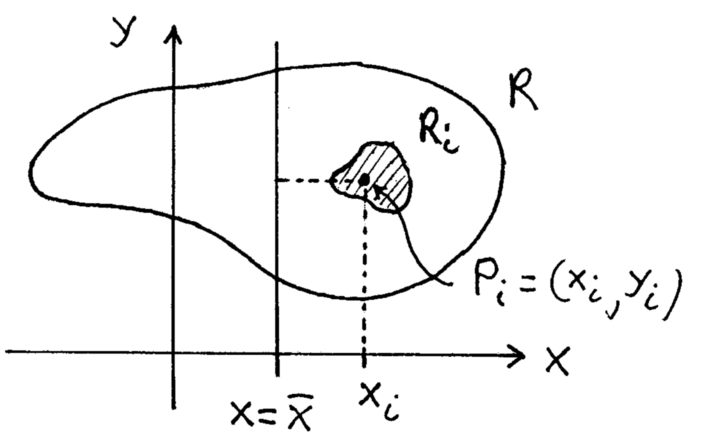
$$\begin{aligned} M_i &= (\text{mass of } R_i)\ (\text{its distance from line } x = \bar{x}) \\ &\approx (\text{area of } R_i)\ (\text{density of } R_i)\ (\text{its distance from line } x = \bar{x}) \\ &\approx (A_i)\ (\delta(P_i))\ (x_i - \bar{x})\ . \end{aligned}$$

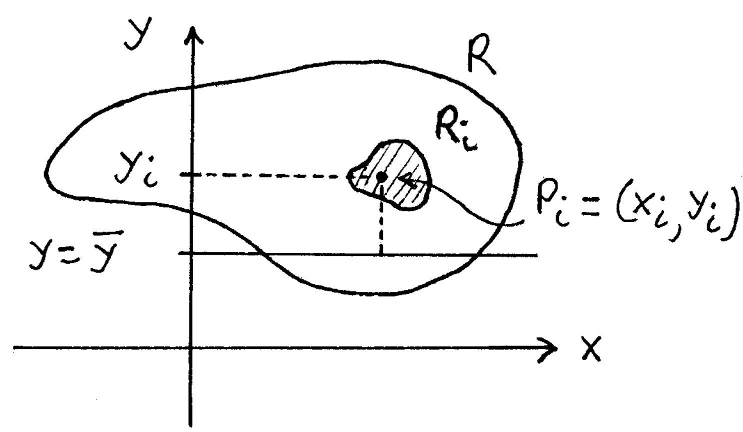
Thus, the total *Moment of R* about the vertical line $x = \bar{x}$ is

$$M_{\bar{x}} = \lim_{mesh \to 0} \sum_{i=1}^{n} (\delta(P_i))(x_i - \bar{x})(A_i) = \int_R \delta(P)(x - \bar{x})\, dA\ .$$

Similarly, the total *Moment of R* about the horizontal line $y = \bar{y}$ is

$$M_{\bar{y}} = \lim_{mesh \to 0} \sum_{i=1}^{n} (\delta(P_i))(y_i - \bar{y})(A_i) = \int_R \delta(P)(y - \bar{y})\, dA\ .$$





To find the center of mass of R we will assume that both $M_{\bar{x}} = 0$ and $M_{\bar{y}} = 0$. If $M_{\bar{x}} = 0$, then

$$\int_R \delta(P)(x - \bar{x})\, dA = 0 \quad \longrightarrow \quad \int_R x \cdot \delta(P)\, dA \ - \ \int_R \bar{x} \cdot \delta(P)\, dA \ = \ 0$$

$$\longrightarrow \quad \int_R x \cdot \delta(P)\, dA \ - \ \bar{x} \int_R \delta(P)\, dA \ = \ 0$$

$$\longrightarrow \quad \bar{x}\int_R \delta(P)\,dA \quad = \quad \int_R x\cdot\delta(P)\,dA$$

so that

$$\bar{x} = \frac{\int_R x\cdot\delta(P)\,dA}{\int_R \delta(P)\,dA}$$

Similarly, if we assume that $M_{\bar{y}} = 0$, then it follows that

$$\bar{y} = \frac{\int_R y\cdot\delta(P)\,dA}{\int_R \delta(P)\,dA}$$