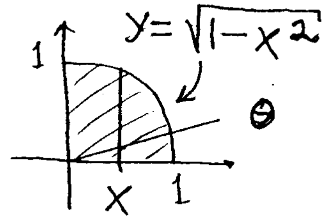
Math 21C Practice Exam 2 Solutions

1.) a.) $\displaystyle\int_0^1 \int_0^{\sqrt{1-x^2}} e^{x^2+y^2}\, dy\, dx$



$$= \int_0^{\frac{\pi}{2}} \int_0^1 e^{r^2} \cdot r \, dr\, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \frac{1}{2} e^{r^2} \Big|_0^1 \, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \frac{1}{2}(e-1)\, d\theta = \frac{\pi}{4}(e-1)$$

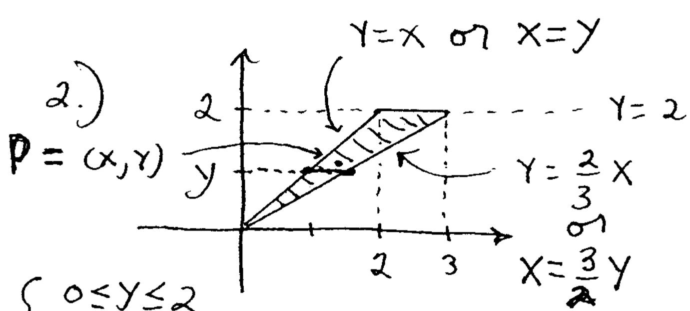
b.) $\displaystyle\int_{\frac{\pi}{4}}^{\pi} \int_{\frac{\pi}{x}}^{2} \int_0^{\cos(xy)} x \, dz\, dy\, dx$

$$= \int_{\frac{\pi}{4}}^{\pi} \int_{\frac{\pi}{x}}^{2} x z \Big|_{z=0}^{z=\cos(xy)} dy\, dx = \int_{\frac{\pi}{4}}^{\pi} \int_{\frac{\pi}{x}}^{2} x \cos(xy)\, dy\, dx$$

$$= \int_{\frac{\pi}{4}}^{\pi} \sin(xy) \Big|_{y=\frac{\pi}{x}}^{y=2} dx = \int_{\frac{\pi}{4}}^{\pi} \sin 2x \, dx$$

$$= \frac{-1}{2} \cos 2x \Big|_{\frac{\pi}{4}}^{\pi} = \frac{-1}{2}(1-0) = \frac{-1}{2}$$

2.)

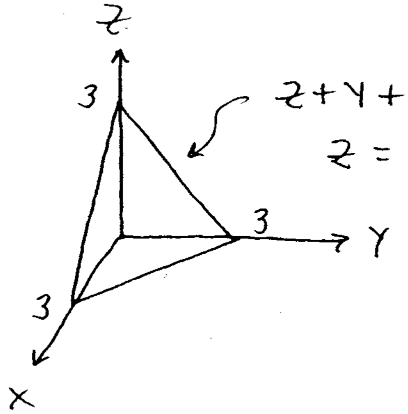


density $e(P) = x^2+y$;
distance from P to line is $\text{dist} = 2-y$;

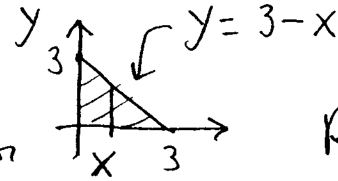
$R: \begin{cases} 0 \le y \le 2 \\ y \le x \le \frac{3}{2}y \end{cases}$

$$\text{M.I.} = \int_R (\text{dist})^2 e(P)\, dA = \int_0^2 \int_y^{\frac{3}{2}y} (2-y)^2 (x^2+y)\, dx\, dy$$

3.)



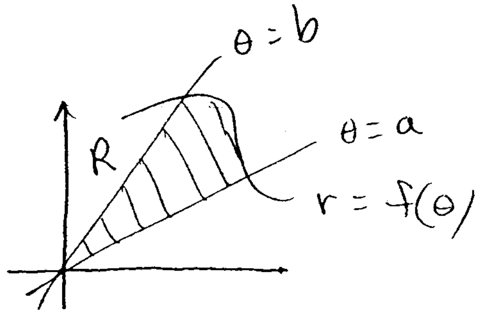
$z+y+x=3$ or $z=3-x-y$



$$R: \begin{cases} 0 \le x \le 3, \\ 0 \le y \le 3-x, \\ 0 \le z \le 3-x-y, \end{cases}$$

$$\text{Vol} = \int_0^3 \int_0^{3-x} \int_0^{3-x-y} 1 \, dz \, dy \, dx$$

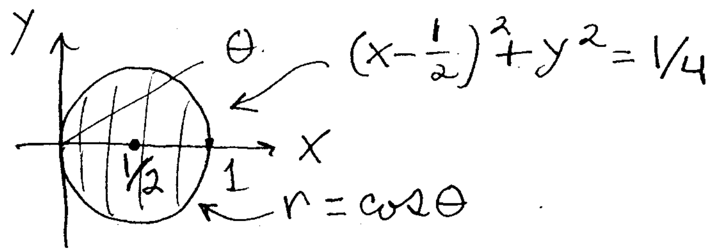
4.)

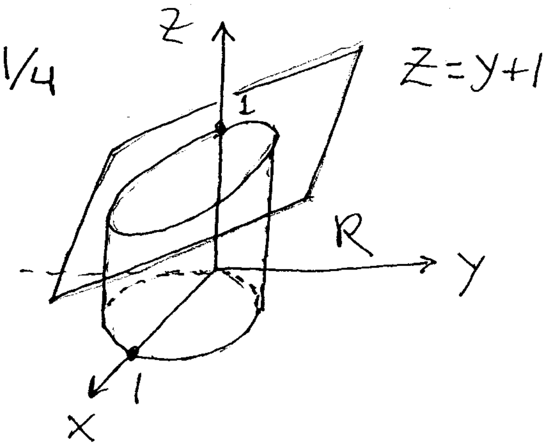


$$\text{area } R = \int_R 1 \, dA = \int_a^b \int_0^{f(\theta)} r \, dr \, d\theta$$

$$= \int_a^b \frac{1}{2} r^2 \Big|_0^{f(\theta)} d\theta = \int_a^b \frac{1}{2} \left(f(\theta)\right)^2 d\theta.$$

5.)





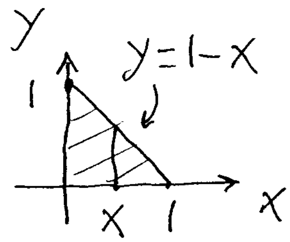
$$R: \begin{cases} -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}, \\ 0 \le r \le \cos\theta \\ 0 \le z \le r\sin\theta + 1 \end{cases}$$

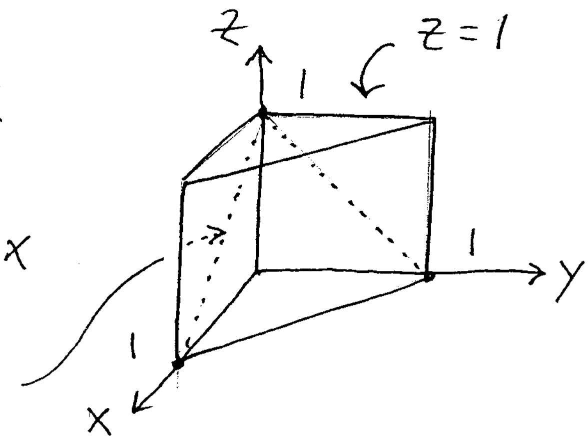
$$\text{Vol. of } R = \int_R 1 \, dV = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{\cos\theta} \int_0^{r\sin\theta+1} 1 \cdot r \, dz \, dr \, d\theta,$$

$$\bar{y} = \frac{\int_R y\,dV}{\text{Vol. of } R} = \frac{1}{\text{Vol. } R}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{\cos 2\theta}\int_0^{r\sin\theta+1}(r\sin\theta)\cdot r\,dz\,dr\,d\theta.$$

6.)

$$R: \begin{cases} 0 \le x \le 1 \\ 0 \le y \le 1-x \\ 1-x-y \le z \le 1 \end{cases}$$





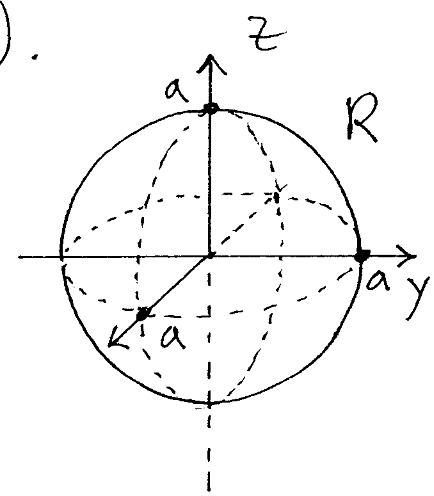
$$\int_0^1\int_0^{1-x}\int_{1-x-y}^{1} 1\,dz\,dy\,dx = \int_0^1\int_0^{1-x}\left(z\Big|_{1-x-y}^{1}\right)dy\,dx$$

$$= \int_0^1\int_0^{1-x}(1-(1-x-y))\,dy\,dx = \int_0^1\int_0^{1-x}(x+y)\,dy\,dx$$

$$= \int_0^1 \left(xy+\tfrac{1}{2}y^2\right)\Big|_{y=0}^{y=1-x} dx = \int_0^1\left(x(1-x)+\tfrac{1}{2}(1-x)^2\right)dx$$

$$= \int_0^1\left(x-x^2+\tfrac{1}{2}(x^2-2x+1)\right)dx = \int_0^1\left(\tfrac{1}{2}-\tfrac{1}{2}x^2\right)dx$$

$$= \left(\tfrac{1}{2}x-\tfrac{1}{6}x^3\right)\Big|_0^1 = \tfrac{1}{2}-\tfrac{1}{6} = \tfrac{1}{3}.$$



7.) Vol. of $R = \int_R 1\,dV$

$$= \int_0^{2\pi}\int_0^{\pi}\int_0^{a} 1\cdot \rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi} \frac{\rho^3}{3}\sin\phi\Big|_{\rho=0}^{\rho=a} d\phi\,d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi} \frac{a^3}{3}\sin\phi\,d\phi\,d\theta$$

$$= \frac{a^3}{3}\left(\int_0^{2\pi} -\cos\phi \Big|_0^{\pi}\right) d\theta$$

$$= \frac{a^3}{3}\int_0^{2\pi} \left(-(-1) - -(1)\right) d\theta$$

$$= \frac{a^3}{3}\int_0^{2\pi} 2\, d\theta$$

$$= \frac{a^3}{3} \cdot 2\theta \Big|_0^{2\pi}$$

$$= \frac{4}{3}\pi a^3 .$$