Math 21C (Kouba)

Practice Exam 3 Solutions

1.) a.) $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{2^{n+2}}{(n+1)!} \cdot \frac{n!}{2^{n+1}}$

$= \lim_{n\to\infty} \frac{2}{n+1} = 0 < 1$, so $\sum_{n=1}^{\infty} \frac{2^{n+1}}{n!}$ converges by the ratio test.

b.) $\lim_{n\to\infty} \frac{3^n}{2^n+3^n} = \lim_{n\to\infty} \frac{1}{\frac{2^n}{3^n}+1} = \lim_{n\to\infty} \frac{1}{\left(\frac{2}{3}\right)^n+1}$

$= \frac{1}{0+1} = 1 \neq 0$, so $\sum_{n=2}^{\infty} \frac{3^n}{2^n+3^n}$ diverges by the nth term test.

c.) $\sum_{n=1}^{\infty} \frac{1+2(-1)^n}{n^2} = \frac{-1}{1^2} + \frac{3}{2^2} + \frac{-1}{3^2} + \frac{3}{4^2} + \frac{-1}{5^2} + \cdots$;

$\sum_{n=1}^{\infty} \left| \frac{1+2(-1)^n}{n^2} \right| = \frac{1}{1^2} + \frac{3}{2^2} + \frac{1}{3^2} + \frac{3}{4^2} + \cdots \leq \sum_{n=1}^{\infty} \frac{3}{n^2}$,

which is a convergent p-series $(p=2>1)$; then $\sum_{n=1}^{\infty} \left| \frac{1+2(-1)^n}{n^2} \right|$ converges by the comparison test, so that $\sum_{n=1}^{\infty} \frac{1+2(-1)^n}{n^2}$ converges by the absolute convergence test.

d.) $\lim_{n\to\infty} \frac{n^2-1}{n^2+4} = \lim_{n\to\infty} \frac{1-1/n^2}{1+4/n^2} = \frac{1-0}{1+0} = 1 \neq 0$,

so $\sum_{n=2}^{\infty} \frac{n^2-1}{n^2+4}$ diverges by the nth term test.

e.) $f(x) = \frac{1}{x\sqrt{\ln x}}$ is $+$, $\downarrow$, and continuous for $x \geq 3$, and $\int_3^{\infty} \frac{1}{x\sqrt{\ln x}}\,dx = 2\sqrt{\ln x}\,\Big|_3^{\infty}$

$= \infty$, so that $\displaystyle\sum_{n=3}^{\infty} \frac{1}{n\sqrt{\ln n}}$ diverges by the integral test.

f.) $$\lim_{n\to\infty} \frac{\dfrac{n^3+2n^2-1}{4n^4+2n+200}}{1/n} = \lim_{n\to\infty} \frac{n^4+2n^3-n}{4n^4+2n+200}$$

$$= \lim_{n\to\infty} \frac{1+2/n-1/n^3}{4+2/n^3+200/n^4} = \frac{1}{4} > 0,$$ and $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, so $\displaystyle\sum_{n=1}^{\infty} \frac{n^3+2n^2-1}{4n^4+2n+200}$ diverges by the limit comparison test.

g.) $$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+1)^6}{8^{n+2}} \cdot \frac{8^{n+1}}{n^6}$$

$$= \lim_{n\to\infty} \frac{1}{8}\left(\frac{n+1}{n}\right)^6 = \lim_{n\to\infty} \frac{1}{8}\cdot\left(1+\frac{1}{n}\right)^6 = \frac{1}{8}(1)^6 = \frac{1}{8} < 1,$$

so $\displaystyle\sum_{n=1}^{\infty} \frac{n^6}{8^{n+1}}$ converges by the ratio test.

h.) $f(x) = \dfrac{x}{(x+5)^2}$ is + and continuous for ≥ 1;

$f \downarrow$? : $$f'(x) = \frac{(x+5)^2(1) - x\cdot 2(x+5)}{(x+5)^4} = \frac{5-x}{(x+5)^3}$$ so

$+ \quad 0 \quad -$ f'

$x=5$

f is $\downarrow$ for $x \geq 5$; then $\displaystyle\sum_{n=5}^{\infty} (-1)^{n+1} \frac{n}{(n+5)^2}$ converges by the alternating series test, so that $\displaystyle\sum_{n=2}^{\infty} (-1)^{n+1} \frac{n}{(n+5)^2}$ converges.

i.) $\displaystyle\sum_{n=2}^{\infty} \frac{3}{n^{\sqrt{3}}}$ is a convergent p-series since $p = \sqrt{3} > 1$.

j.) $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(2(n+2))!}{(3(n+1))!} \cdot \frac{(3n)!}{(2(n+1))!}$

$= \lim_{n\to\infty} \frac{(2n+4)!}{(2n+2)!} \cdot \frac{(3n)!}{(3n+3)!}$

$= \lim_{n\to\infty} \frac{(2n+4)(2n+3)}{(3n+3)(3n+2)(3n+1)} \cdot \frac{\frac{1}{n}\cdot\frac{1}{n}}{\frac{1}{n}\cdot\frac{1}{n}}$

$= \lim_{n\to\infty} \frac{(2+4/n)(2+3/n)}{(3+3/n)(3+2/n)(3n+1)} = \frac{4}{\infty} = 0 < 1$

so $\sum_{j=2}^{\infty} \frac{(2(n+1))!}{(3n)!}$ converges by the ratio test.

k.) $\sum_{n=2}^{\infty} (1.1)^n$ diverges by the geometric series since $r = 1.1$ and $|r| \geq 1$; $\sum_{n=2}^{\infty} (0.9)^n$ converges by the geometric series test since $r = 0.9$ and $-1 < r < 1$; thus $\sum_{n=2}^{\infty} ((1.1)^n + (0.9)^n)$ diverges.

2.) By equation (*)(*) the error

$R_n = \frac{1}{(n+1)^2+4} + \frac{1}{(n+2)^2+4} + \frac{1}{(n+3)^2+4} + \cdots < \int_n^{\infty} \frac{1}{x^2+2^2}\,dx$

$= \lim_{A\to\infty} \frac{1}{2}\arctan\left(\frac{x}{2}\right)\Big|_n^A = \lim_{A\to\infty} \frac{1}{2}\left(\arctan\left(\frac{A}{2}\right) - \arctan\left(\frac{n}{2}\right)\right)$

$= \frac{1}{2}\left(\frac{\pi}{2} - \arctan\left(\frac{n}{2}\right)\right) \leq 0.00001 \Rightarrow$

$\arctan\left(\frac{n}{2}\right) \geq \frac{\pi}{2} - 0.00002 \Rightarrow$

$\frac{n}{2} \geq \tan\left(\frac{\pi}{2} - 0.00002\right) \Rightarrow$

$n \geq 2\tan\left(\frac{\pi}{2} - 0.00002\right) \approx 99999.99$

so choose $n \geq 100{,}000$.

3.) By equation (*) $\int_1^{n+1} \frac{1}{x}\,dx < 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$;

choose n so that

$$200 \leq \int_1^{n+1} \frac{1}{x}\,dx = \ln x \Big|_1^{n+1} = \ln(n+1) - \cancel{\ln 1}^{\,0} \Rightarrow$$

$$\ln(n+1) \geq 200 \Rightarrow e^{\ln(n+1)} \geq e^{200} \Rightarrow$$

$$n+1 \geq e^{200} \Rightarrow n \geq e^{200} - 1 \approx 7.2 \times 10^{86}.$$

4.) $\frac{4}{9} - \frac{8}{27} + \frac{16}{81} - \frac{32}{243} + \frac{64}{729} - \cdots$

$$= \left(\frac{4}{9}\right) \cdot \left[1 - \frac{2}{3} + \frac{4}{9} - \frac{8}{27} + \frac{16}{81} - \cdots\right]$$

$$= \left(\frac{4}{9}\right) \cdot \left[1 + \left(\frac{-2}{3}\right) + \left(\frac{-2}{3}\right)^2 + \left(\frac{-2}{3}\right)^3 + \cdots\right]$$

$$= \left(\frac{4}{9}\right) \cdot \frac{1}{1 - \left(\frac{-2}{3}\right)} = \left(\frac{4}{9}\right)\left(\frac{3}{5}\right) = \frac{4}{15}$$

5.) $0.343434\cdots = \frac{34}{10^2} + \frac{34}{10^4} + \frac{34}{10^6} + \frac{34}{10^8} + \cdots$

$$= \left(\frac{34}{10^2}\right)\left[1 + \frac{1}{10^2} + \frac{1}{10^4} + \frac{1}{10^6} + \cdots\right]$$

$$= \left(\frac{34}{100}\right) \cdot \left[1 + \left(\frac{1}{100}\right) + \left(\frac{1}{100}\right)^2 + \left(\frac{1}{100}\right)^3 + \cdots\right]$$

$$= \left(\frac{34}{100}\right) \cdot \frac{1}{1 - \left(\frac{1}{100}\right)} = \left(\frac{34}{100}\right)\left(\frac{100}{99}\right) = \frac{34}{99}.$$