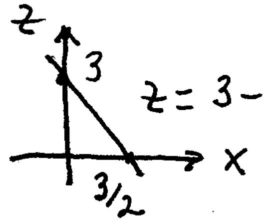
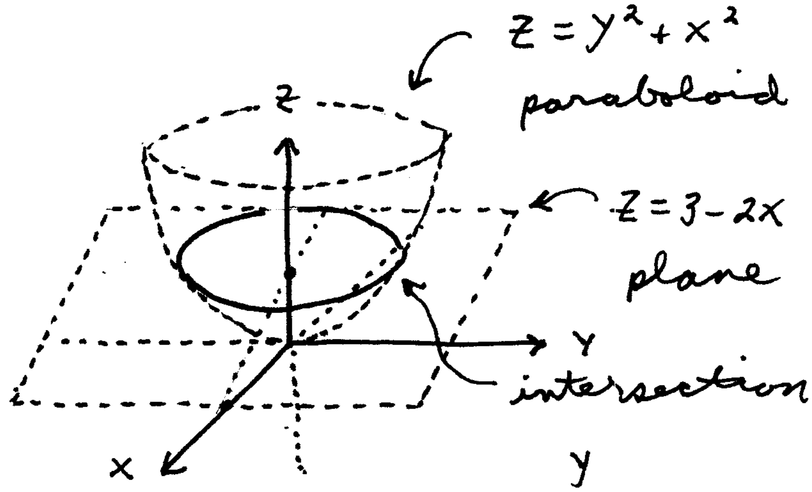
Math 21C
Exam 1 Solutions

1.)





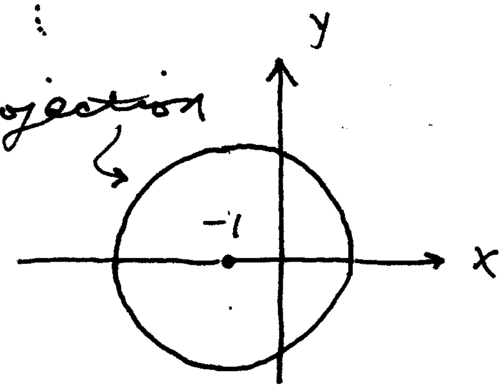
$z = y^2 + x^2$, $z = 3 - 2x$

$\rightarrow\ 3 - 2x = y^2 + x^2$

$\rightarrow\ 3 + 1 = y^2 + x^2 + 2x + 1$

$\rightarrow\ 4 = y^2 + (x+1)^2$

circle: $r = 2$, center: $(-1, 0)$



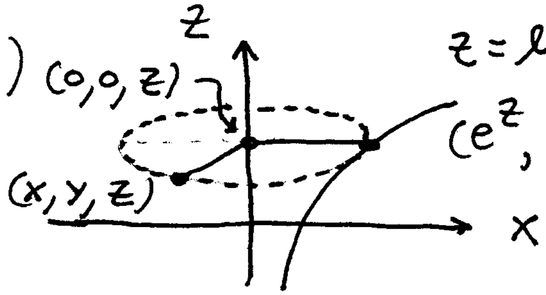
2.) $x^2 + 2x + y^2 - 6y + z^2 = -6 \rightarrow$

$x^2 + 2x + 1 + y^2 - 6y + 9 + z^2 = -6 + 1 + 9 = 4 \rightarrow$

$(x+1)^2 + (y-3)^2 + (z-0)^2 = 2^2 \rightarrow$

radius $= 2$, center: $(-1, 3, 0)$

3.)



$z = \ln x$ or $x = e^z$

$(e^z, 0, z)$ $\quad \sqrt{(x-0)^2 + (y-0)^2 + (z-z)^2}$

$= \sqrt{(e^z - 0)^2 + (0-0)^2 + (z-z)^2}$

$\rightarrow\ x^2 + y^2 = (e^z)^2 = e^{2z}$

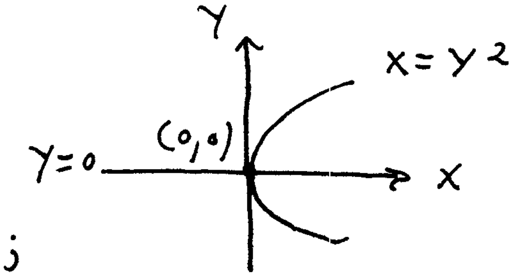
4.) a.) $\displaystyle \lim_{(x,y)\to(3,-3)} \frac{x^4 - y^4}{x+y} \overset{0}{\underset{0}{=}} \lim_{(x,y)\to(3,-3)} \frac{(x-y)\cancel{(x+y)}(x^2+y^2)}{\cancel{x+y}}$

$= (6)(18) = 108$

b.) <u>along $y=0$</u>:

$\displaystyle \lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^4} = \lim_{x\to 0} \frac{0}{x^2}$

$\displaystyle = \lim_{x\to 0} 0 = 0$;



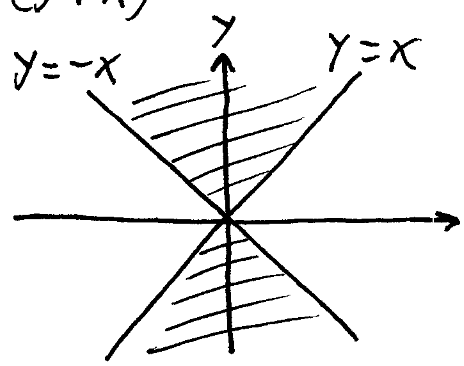
along $x = y^2$: $\lim_{(x,y)\to(0,0)} \frac{y^4}{y^4 + y^4} = \lim_{(x,y)\to(0,0)} \frac{y^4}{2y^4}$

$= \lim_{(x,y)\to(0,0)} \frac{1}{2} = \frac{1}{2}$; thus

$\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^4}$ DNE.

5.) $f(x,y) = \sqrt{y^2 - x^2} = \sqrt{(y-x)(y+x)}$

Domain of f:
lines $y = x$, $y = -x$
and shaded
region

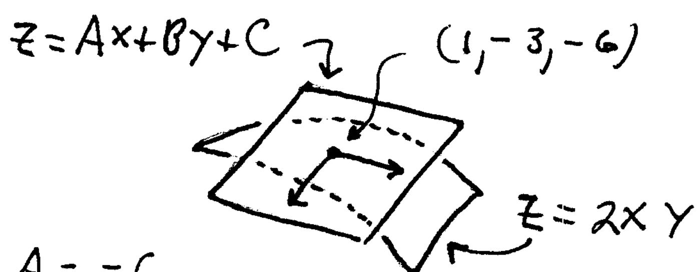


6.) $f(x,y) = \ln(x^2+y^2) \rightarrow f_x = \frac{2x}{x^2+y^2} \rightarrow$

$f_{xx} = \frac{(x^2+y^2)(2) - 2x\cdot(2x)}{(x^2+y^2)^2} = \frac{2y^2 - 2x^2}{(x^2+y^2)^2}$,

$f_y = \frac{2y}{x^2+y^2} \rightarrow f_{yy} = \frac{(x^2+y^2)(2) - 2y\cdot(2y)}{(x^2+y^2)^2} = \frac{2x^2 - 2y^2}{(x^2+y^2)^2}$

then $f_{xx} + f_{yy} = \frac{2y^2 - 2x^2 + 2x^2 - 2y^2}{(x^2+y^2)^2} = \frac{0}{(x^2+y^2)^2} = 0$.



7.) $z = 2xy \rightarrow$

$z_x = 2y = 2(-3) = -6$,

$z_y = 2x = 2(1) = 2$;

$z = Ax + By + C \rightarrow z_x = A = -6$,

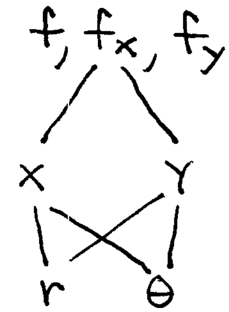
$z_y = B = 2 \rightarrow z = -6x + 2y + C \rightarrow$

$-6 = -6(1) + 2(-3) + c \to c = 6 \to$
tangent plane $z = -6x + 2y + 6$.

8.) $z = x^3 + 3xy - y^3 \to z_x = 3x^2 + 3y = 0 \to$
$\underline{y = -x^2}$, $z_y = 3x - 3y^2 = 0 \to \underline{x = y^2}$; then
$y = -x^2 = -(y^2)^2 \to y^4 + y = 0 \to y(y^3 + 1) = 0$
$\to y = 0$ or $y = -1$; if $y =$ $\to x = 0$ so
$\boxed{(0,0)}$ is crit. pt., if $y = -1 \to x = 1$ so $\boxed{(1,-1)}$
is crit. pt. ; $z_{xx} = 6x$, $z_{yy} = -6y$, $z_{xy} = 3$

$\underline{\underline{\text{For } (0,0)}}$: $D = z_{xx} z_{yy} - (z_{xy})^2 = (0)(0) - (3)^2 = -9 < 0$
so $(0,0)$ determines a saddle point.
$\underline{\underline{\text{For } (1,-1)}}$: $D = z_{xx} z_{yy} - (z_{xy})^2 = (6)(6) - (3)^2 = 27 > 0$
and $z_{xx} = 6 > 0$ so $(1,-1)$ determines a relative minimum value of $z = -1$.

9.) $z = f(x,y)$, $x = r\cos\theta$, $y = r\sin\theta \to$

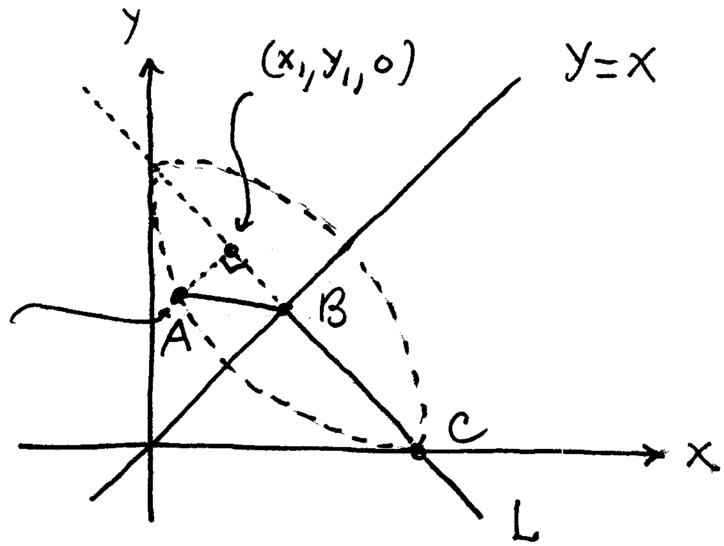


$\frac{\partial z}{\partial \theta} = f_x \cdot \frac{\partial x}{\partial \theta} + f_y \cdot \frac{\partial y}{\partial \theta} = f_x \cdot -r\sin\theta$
$+ f_y \cdot r\cos\theta$,

$\frac{\partial^2 z}{\partial \theta^2} = -r \cdot [f_x \cdot \cos\theta +$
$\sin\theta \cdot (f_{xx} \cdot \frac{\partial x}{\partial \theta} + f_{xy} \cdot \frac{\partial y}{\partial \theta})] + r[f_y \cdot -\sin\theta$
$+ \cos\theta \cdot (f_{yx} \cdot \frac{\partial x}{\partial \theta} + f_{yy} \cdot \frac{\partial y}{\partial \theta})$

$= -r\,[f_x \cos\theta + \sin\theta(f_{xx} \cdot -r\sin\theta + f_{xy} \cdot r\cos\theta)]$
$+ r\,[-f_y \cdot \sin\theta + \cos\theta(f_{xy} \cdot -r\sin\theta + f_{yy} \cdot r\cos\theta]$

Extra Credit:



Let (x_1, y_1, z_1) be a random point on the surface. Then pt. (x_1, y_1) lies on line L. The equation of line L is $\underline{y = -x + (x_1 + y_1)}$; find $\cap$ with line $y = x$:

$$x = -x + (x_1 + y_1) \rightarrow \quad x = \tfrac{1}{2}(x_1 + y_1), \; y = \tfrac{1}{2}(x_1 + y_1);$$

then $A = (x_1, y_1, z_1)$,

$B = \left(\tfrac{1}{2}(x_1 + y_1), \tfrac{1}{2}(x_1 + y_1), 0\right)$,

$C = (x_1 + y_1, 0, 0)$; set $AB = BC$:

$$\sqrt{\left(\tfrac{1}{2}(x_1 - y_1)\right)^2 + \left(\tfrac{1}{2}(y_1 - x_1)\right)^2 + (z_1 - 0)^2}$$

$$= \sqrt{\left(-\tfrac{1}{2}(x_1 + y_1)\right)^2 + \left(\tfrac{1}{2}(x_1 + y_1)\right)^2 + (0 - 0)^2} \rightarrow$$

$$\tfrac{1}{4}(x_1 - y_1)^2 + \tfrac{1}{4}(x_1 - y_1)^2 + z_1^2 = \tfrac{1}{4}(x_1 + y_1)^2 + \tfrac{1}{4}(x_1 + y_1)^2 \rightarrow$$

$$\tfrac{1}{2}(x_1 - y_1)^2 + z_1^2 = \tfrac{1}{2}(x_1 + y_1)^2 \rightarrow$$

$$-x_1 y_1 + z_1^2 = x_1 y_1 \rightarrow$$

$$z_1^2 = 2 x_1 y_1 \quad \text{so}$$

surface has equation

$$z^2 = 2xy .$$