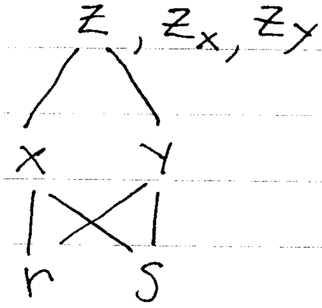
# Math 21C
## Exam 2 Solutions




1.) $\frac{\partial z}{\partial r} = z_x \cdot \frac{\partial x}{\partial r} + z_y \cdot \frac{\partial y}{\partial r}$

$= z_x \cdot 2rs + z_y \cdot 3$ ,

$\frac{\partial^2 z}{\partial r^2} = \frac{\partial}{\partial r}\left(\frac{\partial z}{\partial r}\right) = \frac{\partial}{\partial r}\left(z_x \cdot 2rs + 3 z_y\right)$

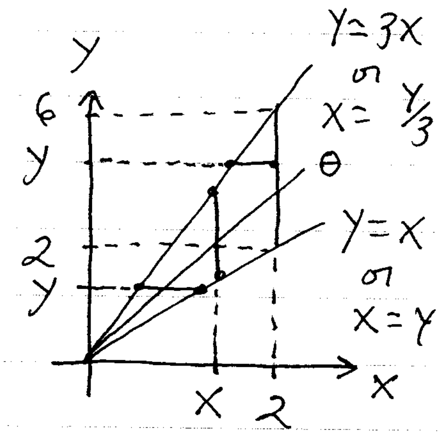
$= z_x \cdot \frac{\partial}{\partial r}(2rs) + \frac{\partial}{\partial r}(z_x) \cdot 2rs + 3 \cdot \frac{\partial}{\partial r}(z_y)$

$= z_x \cdot 2s + \left(z_{xx} \cdot \frac{\partial x}{\partial r} + z_{xy} \cdot \frac{\partial y}{\partial r}\right) \cdot 2rs$

$+ 3\left(z_{yx} \cdot \frac{\partial x}{\partial r} + z_{yy} \cdot \frac{\partial y}{\partial r}\right)$

$= 2s \cdot z_x + \left(z_{xx} \cdot 2rs + z_{xy} \cdot 3\right) \cdot 2rs$

$+ 3\left(z_{xy} \cdot 2rs + z_{yy} \cdot 3\right)$




2.) $x = 2 \rightarrow r\cos\theta = 2 \rightarrow$

$r = 2\sec\theta$

a.) $0 \le x \le 2$ and $x \le y \le 3x$

b.) $0 \le y \le 2$ and $y/3 \le x \le y$,

$2 \le y \le 6$ and $y/3 \le x \le 2$

c.) $\frac{\pi}{4} \le \theta \le \arctan 3$ and $0 \le r \le 2\sec\theta$

3.) a.) $\int_0^{\frac{\pi}{2}} \int_0^x 2\cos(x+y)\,dy\,dx$

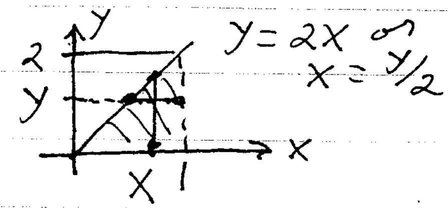
$= \int_0^{\frac{\pi}{2}} \left(2\sin(x+y)\Big|_{y=0}^{y=x}\right)dx = \int_0^{\frac{\pi}{2}} (2\sin 2x - 2\sin x)\,dx$

$= \left(-\cos 2x + 2\cos x\right)\Big|_0^{\pi/2} = \left(-\cos\pi + 2\overset{0}{\cancel{\cos\frac{\pi}{2}}}\right)$

$- \left(-\cos 0 + 2\cos 0\right) = -(-1) - (-1+2) = 0$

b.) $\int_0^2 \int_{y/2}^1 e^{x^2}\,dx\,dy$

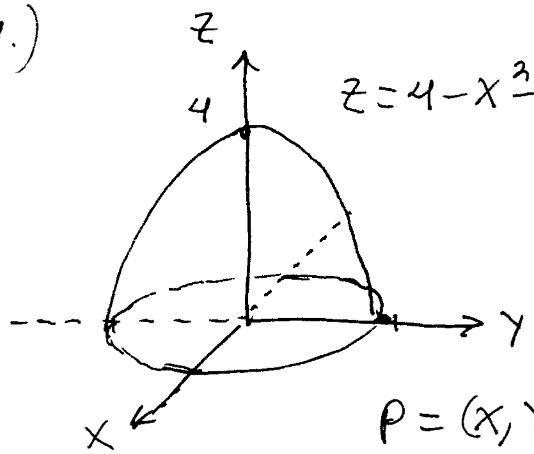
(switch order)

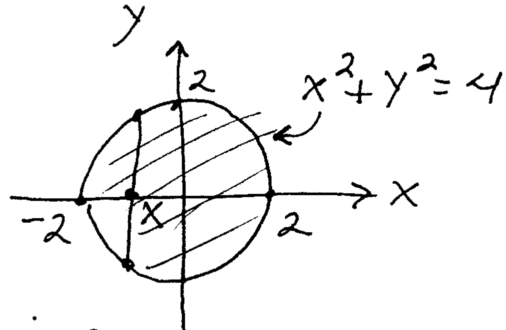


$$= \int_0^1 \int_0^{2x} e^{x^2}\,dy\,dx = \int_0^1 \left( y e^{x^2}\Big|_{y=0}^{y=2x} \right) dx$$

$$= \int_0^1 2xe^{x^2}\,dx = e^{x^2}\Big|_0^1 = e^1 - e^0 = e - 1 .$$

4.)

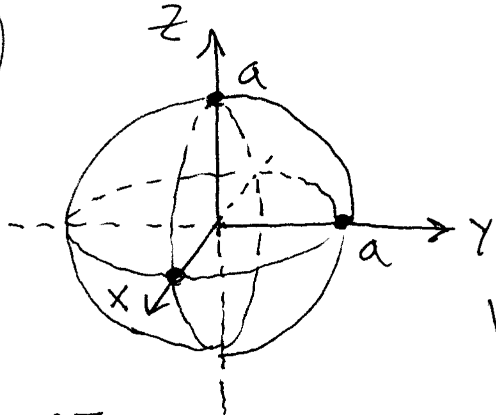



density at pt. $P = (x, y, z)$ is $f(P) = z$.

$$\text{Mass of } R = \int_R f(P)\,dV = \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_0^{4-x^2-y^2} z\,dz\,dy\,dx$$

5.)




$$R: \begin{cases} 0 \le \theta \le 2\pi, \\ 0 \le \phi \le \pi, \\ 0 \le \rho \le a \end{cases},$$

$$\text{Vol} = \int_R 1\,dV$$

$$= \int_0^{2\pi} \int_0^{\pi} \int_0^{a} \rho^2 \sin\phi\,d\rho\,d\phi\,d\theta = \int_0^{2\pi} \int_0^{\pi} \left( \frac{1}{3}\rho^3 \sin\phi \Big|_{\rho=0}^{\rho=a} \right) d\phi\,d\theta$$

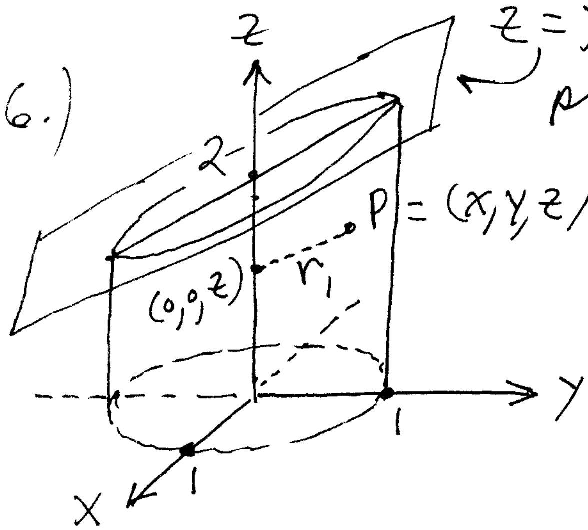
$$= \frac{1}{3}a^3 \int_0^{2\pi} \int_0^{\pi} \sin\phi\,d\phi\,d\theta = \frac{1}{3}a^3 \int_0^{2\pi} \left( -\cos\phi \Big|_{\phi=0}^{\phi=\pi} \right) d\theta$$

$$= \frac{1}{3}a^3 \int_0^{2\pi} (-\cos\pi - -\cos 0)\,d\theta$$

$$= \frac{1}{3}a^3 \int_0^{2\pi} 2\,d\theta = \frac{1}{3}a^3\, 2\theta \Big|_0^{2\pi}$$

$$= \frac{4}{3}\pi a^3 .$$

6.)




$z = y+2$ plane

density $f(x,y,z) = x^2 z$ ;

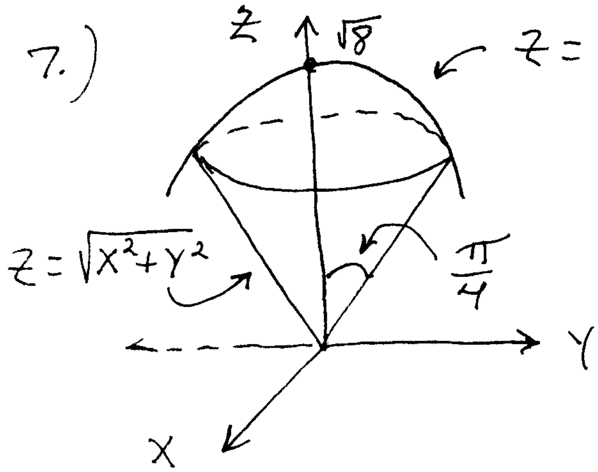
distance from P to z-axis is

$$r_1 = \sqrt{(x-0)^2+(y-0)^2+(z-z)^2}$$
$$= \sqrt{x^2+y^2}$$
$$= \sqrt{r^2} = r \ ;$$

$$R: \begin{cases} 0 \le \theta \le 2\pi, \\ 0 \le r \le 1, \\ 0 \le z \le r\sin\theta + 2 \end{cases} ; \qquad \text{M. of I.} = \int_R r_1^2 \cdot f(P)\, dV$$

$$= \int_0^{2\pi}\int_0^1\int_0^{r\sin\theta+2} r^2 \cdot (r\cos\theta)^2 z \cdot r\, dz\, dr\, d\theta \ .$$

7.)




$z = \sqrt{8-x^2-y^2}$ or $\rho = \sqrt{8}$ ,

$f(x,y,z) = yz$ ;

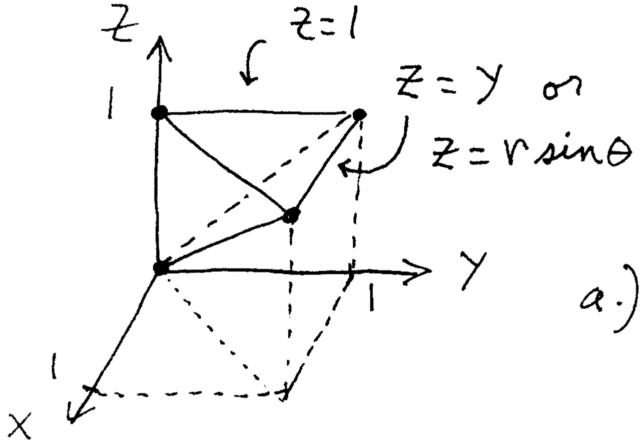
$$AVE = \frac{1}{Vol\,R} \int_R f(P)\, dV \ ;$$

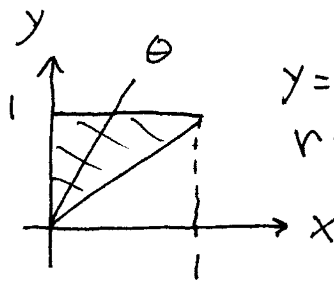
$$Vol\,R = \int_0^{2\pi}\int_0^{\frac{\pi}{4}}\int_0^{\sqrt{8}} 1 \cdot \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta ,$$

$$R: \begin{cases} 0 \le \theta \le 2\pi, \\ 0 \le \phi \le \frac{\pi}{4}, \\ 0 \le \rho \le \sqrt{8} \end{cases}$$

$$AVE = \frac{1}{Vol\,R} \cdot \int_0^{2\pi}\int_0^{\frac{\pi}{4}}\int_0^{\sqrt{8}} (\rho\sin\phi\sin\theta)(\rho\cos\phi)\, \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$$

## Extra Credit:

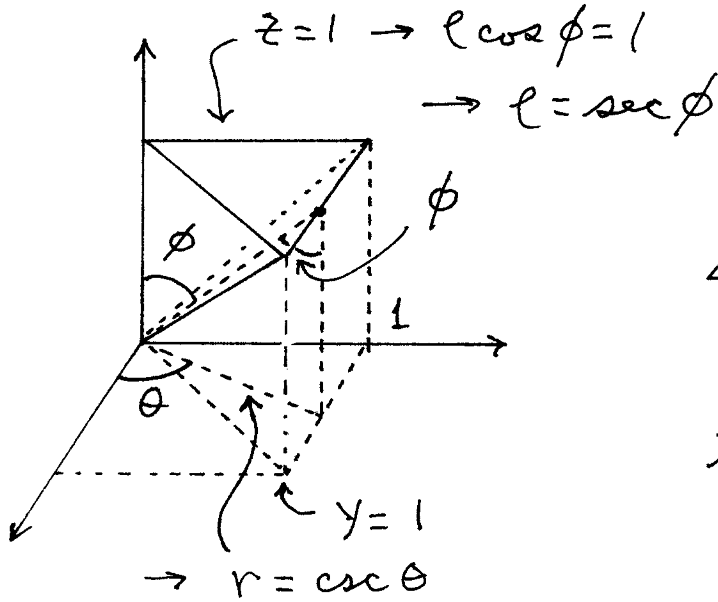


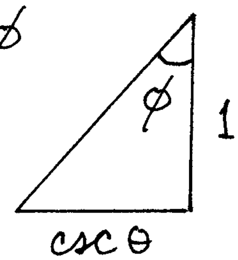




$y = 1 \to r = \csc\theta$

a.) $R: \begin{cases} \frac{\pi}{4} \le \theta \le \frac{\pi}{2}, \\ 0 \le r \le \csc\theta, \\ r\sin\theta \le z \le 1 \end{cases}$

b.)




$\tan\phi = \dfrac{\csc\theta}{1} \to$

$\phi = \arctan(\csc\theta)$

$R: \begin{cases} \frac{\pi}{4} \le \theta \le \frac{\pi}{2}, \\ 0 \le \phi \le \arctan(\csc\theta), \\ 0 \le \rho \le \sec\phi \end{cases}$