Math 21 C

Exam 3 Solutions

1.) a.) $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{2^{n+2}}{n!} \cdot \frac{(n-1)!}{2^{n+1}}$

$= \lim_{n\to\infty} \frac{2}{n} = 0 < 1$, so $\sum_{n=1}^{\infty} \frac{2^{n+1}}{(n-1)!}$ converges by ratio test.

b.) $\lim_{n\to\infty} \frac{n+1}{5n+2} = \lim_{n\to\infty} \frac{1+1/n}{5+2/n} = \frac{1}{5} \neq 0$, so $\sum_{n=2}^{\infty} \frac{n+1}{5n+2}$ diverges by the nth term test.

c.) $\lim_{n\to\infty} \frac{\frac{n+1}{5n^3+2}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{n^3+n^2}{5n^3+2} = \lim_{n\to\infty} \frac{\cancel{1}+1/n}{5+2/n^3}$

$= \frac{1}{5} > 0$, so $\sum_{n=1}^{\infty} \frac{n+1}{5n^3+2}$ converges by the limit comparison test, since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is a convergent p-series ($p = 2 > 1$).

d.) Let $f(x) = \frac{1}{x(\ln x)^2}$, which is $+$, $\downarrow$, and continuous for $x \geq 2$; then $\int_2^{\infty} \frac{1}{x(\ln x)^2}\,dx$

$= \lim_{A\to\infty} \int_2^A \frac{1}{u^2}\,du = \lim_{A\to\infty} \frac{-1}{u}\Big|_2^A = \lim_{A\to\infty} \frac{-1}{\ln x}\Big|_2^A$

$= \lim_{A\to\infty} \left(\frac{-1}{\ln A} - \frac{-1}{\ln 2}\right) = 0 + \frac{1}{\ln 2} < \infty$, so $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$ converges by the integral test.

e.) $\lim_{n\to\infty} a_n^{1/n} = \lim_{n\to\infty} \left[\left(\frac{n+2}{2n+3}\right)^n\right]^{\frac{1}{n}} = \lim_{n\to\infty} \frac{n+2}{2n+3}$

$= \frac{1}{2} < 1$, so $\sum_{n=1}^{\infty} \left(\frac{n+2}{2n+3}\right)^n$ converges by the root test.

f.) $a_n = \dfrac{1}{n(n^2+n+1)}$ is $+$, $\downarrow$, and $\displaystyle\lim_{n\to\infty} \frac{1}{n(n^2+n+1)} = 0$, so $\displaystyle\sum_{n=2}^{\infty} (-1)^{n+1} \cdot \frac{1}{n(n^2+n+1)}$ converges by the alternating series test.

g.) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{n+1} - \frac{1}{n+2}\right) = \sum_{n=1}^{\infty} \frac{n+2-(n+1)}{(n+1)(n+2)} = \sum_{n=1}^{\infty} \frac{1}{n^2+3n+2}$;

$\displaystyle\lim_{n\to\infty} \frac{\frac{1}{n^2+3n+2}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{n^2}{n^2+3n+2} = \lim_{n\to\infty} \frac{1}{1+\frac{3}{n}+\frac{2}{n^2}}$

$= 1 > 0$, so $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{n+1} - \frac{1}{n+2}\right)$ converges by the limit comparison test since $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ is a convergent p-series ($p = 2 > 1$) ; $\boxed{\text{OR}}$

$S_1 = \frac{1}{2} - \frac{1}{3}$, $S_2 = \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) = \frac{1}{2} - \frac{1}{4}$,

$S_3 = \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{4} - \frac{1}{5}\right) = \frac{1}{2} - \frac{1}{5}$, ...,

$S_n = \frac{1}{2} - \frac{1}{n+2}$ and $\displaystyle\lim_{n\to\infty} S_n = \frac{1}{2}$, so $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{n+1} - \frac{1}{n+2}\right)$ converges by the sequence of partial sums test.

h.) $\displaystyle\lim_{n\to\infty} \frac{\frac{\ln n}{n^2}}{\frac{1}{n^{3/2}}} = \lim_{n\to\infty} \frac{\ln n}{n^2} \cdot \frac{n^{3/2}}{1} = \lim_{n\to\infty} \frac{\ln n}{n^{1/2}}$

$\overset{\text{"}\frac{\infty}{\infty}\text{"}}{=} \displaystyle\lim_{n\to\infty} \frac{\frac{1}{n}}{\frac{1}{2\sqrt{n}}} = \lim_{n\to\infty} \frac{2}{\sqrt{n}} = 0$, so $\displaystyle\sum_{n=2}^{\infty} \frac{\ln n}{n^2}$

converges by the limit comparison test since $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^{3/2}}$ is a convergent p-series ($p = \frac{3}{2} > 1$)

2.) $\displaystyle\sum_{n=1}^{\infty} \frac{3^{n+2}}{4^{n-1}} = \sum_{n=1}^{\infty} \frac{3^2 \cdot 3^n}{4^{-1} \cdot 4^n} = 36 \sum_{n=1}^{\infty} \left(\frac{3}{4}\right)^n$

$= 36 \left(\left(\frac{3}{4}\right) + \left(\frac{3}{4}\right)^2 + \left(\frac{3}{4}\right)^3 + \left(\frac{3}{4}\right)^4 + \cdots \right)$

$$= (36)\left(\frac{3}{4}\right)\left(1+\left(\frac{3}{4}\right)+\left(\frac{3}{4}\right)^2+\left(\frac{3}{4}\right)^3+\cdots\right)$$

$$= (27)\cdot\frac{1}{1-\frac{3}{4}} = (27)(4) = 108 .$$

3.) $0.09999\cdots = \frac{9}{100}+\frac{9}{1000}+\frac{9}{10,000}+\cdots$

$$= \frac{9}{100}\left(1+\frac{1}{10}+\frac{1}{100}+\frac{1}{1000}+\cdots\right)$$

$$= \frac{9}{100}\left(1+\left(\frac{1}{10}\right)+\left(\frac{1}{10}\right)^2+\left(\frac{1}{10}\right)^3+\cdots\right) = \frac{9}{100}\cdot\frac{1}{1-\frac{1}{10}} = \frac{9}{100}\cdot\frac{10}{9} = \frac{1}{10}$$

4.) a.) $S_5 = 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}$, so error $R_5 = \underbrace{\frac{-1}{6}+\frac{1}{7}}_{-}+\underbrace{\frac{-1}{8}+\frac{1}{9}}_{-}+\cdots < 0$ and S_5 is an <u>over</u> estimate.

b.) $S_n = 1-\frac{1}{2}+\frac{1}{3}-\cdots+(-1)^{n+1}\frac{1}{n}$, and error $R_n = (-1)^{n+2}\frac{1}{n+1}+(-1)^{n+3}\frac{1}{n+2}+\cdots$ satisfies $|R_n| < \frac{1}{n+1} \leq \frac{1}{1000}$, so choose $n \geq 999$.

5.) By (*)(*) $R_n < \int_n^{\infty}\frac{1}{x^2+1}\,dx = \lim_{A\to\infty} \arctan x \Big|_n^A$

$$= \lim_{A\to\infty}(\arctan A - \arctan n) = \frac{\pi}{2} - \arctan n \leq 0.001$$

$$\Rightarrow \arctan n \geq \frac{\pi}{2}-0.001 \Rightarrow \tan(\arctan n) \geq \tan\left(\frac{\pi}{2}-0.001\right)$$

$\Rightarrow n \geq 999.99$ so choose $n \geq 1000$.

<u>Extra Credit</u>: 1.) a.) $\lim_{n\to\infty}\left(\frac{n^2}{n^2+1}\right)^n = \lim_{n\to\infty}\left(\frac{1}{1+\frac{1}{n^2}}\right)^n$

$= \lim_{n\to\infty}\frac{1}{\left[\left(1+\frac{1}{n^2}\right)^{n^2}\right]^{\frac{1}{n}}} = \frac{1}{e^0} = 1 \neq 0$, so series diverges by the nth term test.

b.) $\frac{n^{n+1/n}}{(n+1/n)^n} \geq \frac{n^n}{(n+\frac{1}{n})^n} = \left(\frac{n}{n+1/n}\right)^n = \left(\frac{n^2}{n^2+1}\right)^n$, so series diverges by comparison test since $\sum_{n=1}^{\infty}\left(\frac{n^2}{n^2+1}\right)^n$ diverges.