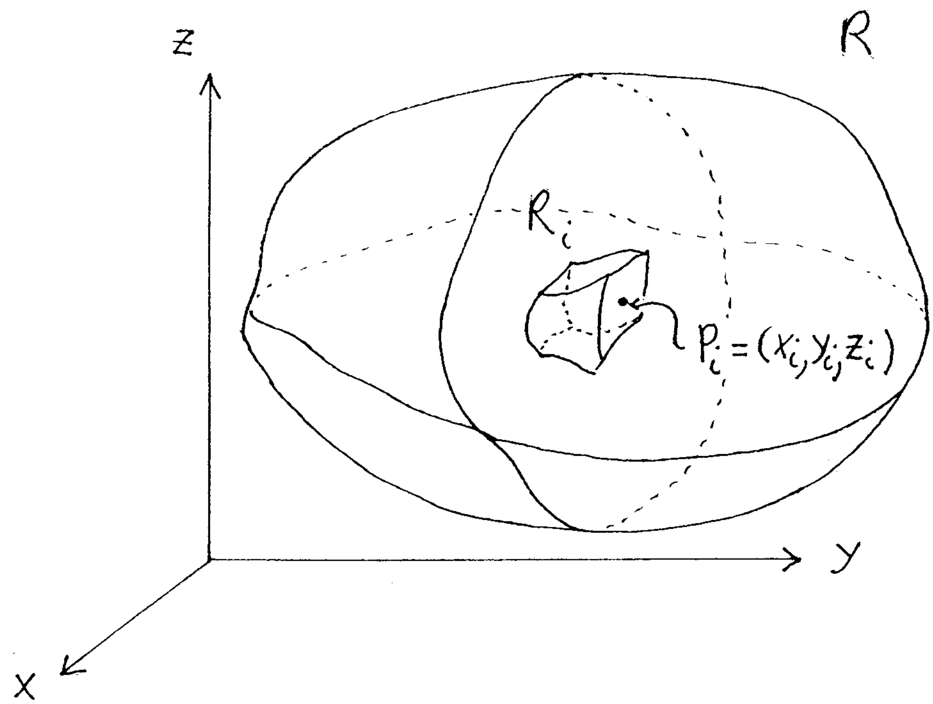
Math 21C
Kouba
Triple Integrals Over Solid Regions R in Three-Dimensional Space
Using Rectangular Coordinates

Consider a solid region R in three-dimensional space and let $w = f(P)$ be a function of three variables defined at each point $P = (x, y, z)$ in R. First partition the solid region R into n parts R_1, R_2, R_3, and R_n of volumes V_1, V_2, V_3, and V_n, resp. Pick sampling point $P_i = (x_i, y_i, z_i)$ in region R_i for $i = 1, 2, 3, \cdots, n$. Define the *diameter* of solid region R_i, $diam(R_i)$, to be the maximum distance between points in R_i for $i = 1, 2, 3, \cdots, n$. Define the *mesh* of the partition to be

$$\text{mesh} = \max_{1 \le i \le n} (diam(R_i)) .$$

Now we define the *integral of f over the solid region R* to be

$$\int_R f(P)\, dV = \lim_{mesh \to 0} \sum_{i=1}^{n} f(P_i) \cdot V_i .$$



In order to motivate the actual evaluation of this integral assume that function $w = f(P)$ represents the density (mass/volume units) at point $P = (x, y, z)$ in R. Then $\int_R f(P)\, dV$ represents the total mass of solid region R. Assume that solid region R is described by

$$a \le x \le b \ , \ g(x) \le y \le k(x) \ , \text{ and } u(x, y) \le z \le v(x, y) .$$

Next let $a = x_0, x_1, x_2, x_3,$ and $x_n = b$ partition the interval $[a, b]$ into n parts. Pick sampling point x_i and let $\Delta x_i = x_i - x_{i-1}$ for $i = 1, 2, 3, \cdots, n$. Define the *mesh* of the partition to be $mesh = \max_{1 \leq i \leq n} (x_i - x_{i-1})$. Make a slice through the solid region R perpendicular to the x-axis at point x_i and let $R(x_i)$ represent the flat intersection of this plane with the solid region R. $R(x_i)$ can be described by

$$g(x_i) \leq y \leq k(x_i) \quad \text{and} \quad u(x_i, y) \leq z \leq v(x_i, y) \ .$$

It follows that

$$\int_{g(x_i)}^{k(x_i)} \int_{u(x_i,y)}^{v(x_i,y)} f(x_i, y, z)\, dz\, dy$$

is a measure in (mass/volume)(area) = (mass/length) units ,

$$\left(\int_{g(x_i)}^{k(x_i)} \int_{u(x_i,y)}^{v(x_i,y)} f(x_i, y, z)\, dz\, dy \right) \Delta x_i$$

is an estimate for the mass of the slice $R(x_i)$ of thickness Δx_i with units (mass), and

$$\lim_{mesh \to 0} \sum_{i=1}^{n} \left(\int_{g(x_i)}^{k(x_i)} \int_{u(x_i,y)}^{v(x_i,y)} f(x_i, y, z)\, dz\, dy \right) \Delta x_i = \int_a^b \int_{g(x)}^{k(x)} \int_{u(x,y)}^{v(x,y)} f(x, y, z)\, dz\, dy\, dx \ .$$

We can now conclude that

$$\int_R f(P)\, dV = \int_a^b \int_{g(x)}^{k(x)} \int_{u(x,y)}^{v(x,y)} f(x, y, z)\, dz\, dy\, dx \ .$$