Math 22A
Kouba
Vector Spaces and Subspaces

DEFINITION: Let V be any set of objects on which two operations are defined: vector addition and scalar multiplication. If all of the following statements are satisfied for all $\vec{u}, \vec{v}, \vec{w} \in V$ and for all scalars $k, l \in R$, then we call V a **vector space** and its object **vectors**.

1.) If $\vec{u}, \vec{v} \in V$, then $\vec{u} \oplus \vec{v} \in V$. (Closure under Vector Addition)

2.) $\vec{u} \oplus \vec{v} = \vec{v} \oplus \vec{u}$. (Commutative Property)

3.) $(\vec{u} \oplus \vec{v}) \oplus \vec{w} = \vec{u} \oplus (\vec{v} \oplus \vec{w})$. (Associative Property)

4.) There is a vector $\vec{0} \in V$ so that $\vec{u} \oplus \vec{0} = \vec{0} \oplus \vec{u} = \vec{u}$. It is called the **zero vector.** (Zero Vector Property)

5.) For each $\vec{u} \in V$ there is a vector $\vec{v} \in V$ so that $\vec{u} \oplus \vec{v} = \vec{v} \oplus \vec{u} = \vec{0}$. We write $\vec{v} = -\vec{u}$ and call it the **negative of** $\vec{u}$. (Additive Inverse Property)

6.) If $k \in R$ and $\vec{u} \in V$, then $k\vec{u} \in V$. (Closure under Scalar Multiplication)

7.) $k(\vec{u} \oplus \vec{v}) = k\vec{u} \oplus k\vec{v}$. (Distributive Property)

8.) $(k + l)\vec{u} = k\vec{u} \oplus l\vec{u}$. (Distributive Property)

9.) $k(l\vec{u}) = (kl)\vec{u}$. (Associative Property)

10.) $1\vec{u} = \vec{u}$. (Scalar Identity Property)

EXAMPLE 1: Consider all n-dimensional vectors in R^n under standard vector addition and scalar multiplication. Properties 1.) through 10.) are satisfied, so R^n is a vector space.

EXAMPLE 2: Let $V = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \, a, b, c, d \in R \right\}$ under standard matrix addition and scalar multiplication. Properties 1.) through 10.) are satisfied, so V is a vector space.

EXAMPLE 3: Let $V = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \, a, b, c, d \in R \; and \; a, b, c, d > 0 \right\}$ under standard matrix addition and scalar multiplication. Properties 1.) through 3.) and 6.) through 10.) are satisfied, but properties 4.) and 5.) fail, so V is NOT a vector space.

EXAMPLE 4: Let $V = \left\{ functions \; f(t) \middle| \, f \; is \; continuous \; for \; all \; values \; of \; t \right\}$ under standard point-wise addition and scalar multiplication. Properties 1.) through 10.) are satisfied, so V is a vector space.

EXAMPLE 5:
Let $V = \left\{ functions\ f(t) \ \middle|\ f\ is\ continuous\ for\ all\ values\ of\ t,\ and\ f(0) = 1 \right\}$ under standard point-wise addition and scalar multiplication. Properties 1.), 4.), 5.), and 6.) are NOT satisfied, so V is NOT a vector space.

EXAMPLE 6: Let $V = \left\{ \overrightarrow{(x, y)} \ \middle|\ x + y = 0 \right\}$ under standard vector addition and scalar multiplication. Properties 1.) through 10.) are satisfied, so V is a vector space.

EXAMPLE 7: Let $V = \left\{ \overrightarrow{(x, y)} \ \middle|\ x + y = 2 \right\}$ under standard vector addition and scalar multiplication. Properties 1.), 4.), 5.), and 6.) are NOT satisfied, so V is NOT a vector space.

DEFINITION: If W is a subset of V and both W and V are vector spaces, then we say W is a **subspace** of V.

EXAMPLE 8: Set V in EXAMPLE 6 is a subspace of R^2.

EXAMPLE 9: The set W of all 2×2 upper triangular matrices is a subspace of the vector space in EXAMPLE 2.

THEOREM: Assume that V is a vector space and W is a subset of V. Then W is a subspace of V iff the following two statements are true.

a.) If $\vec{u}, \vec{v} \in V$, then $\vec{u} \oplus \vec{v} \in V$.

b.) If $k \in R$ and $\vec{u} \in V$, then $k\vec{u} \in V$.