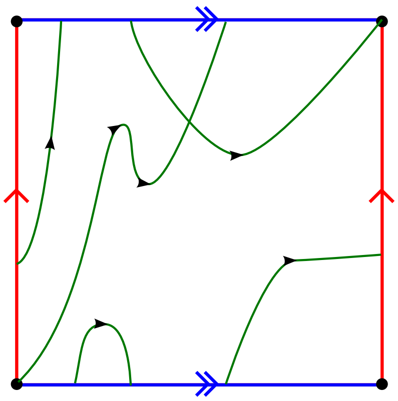
HOMOTOPY

Reading: Lee *Introduction to Topological Manifolds*, Chapter 7

(1) Give the definitions of a homotopy between two paths, a homotopy between two loops, a homotopy between two maps $f_0 : X \to Y$ and $f_1 : X \to Y$, and a homotopy equivalence between two spaces. Give an example for each definition.

(2) Give the definition of $\pi_1(X, p)$. Prove that $\pi_1(\mathbb{R}^n, 0)$ is trivial by giving an explicit method to construct a homotopy from any loop starting and ending at the origin to the constant loop at the origin. If you can try to show that $\pi_k(\mathbb{R}^n, 0)$ is trivial for all k.

(3) Consider the torus T^2 with the square model. Let p be the vertex. The horizontal and vertical edges of the square give two loops based at p. For any other loop in the torus, we count the intersections of that loop with the horizontal and vertical loops with positive and negative signs in the following way. To count the intersections of an arbitrary loop with the horizontal loop, each time it passes through the horizontal edge going upward through the top edge add $+1$ to the count, and each time it passes through the horizontal edge going downward through the bottom edge add -1 to the count. Similarly for the intersection number with the vertical edge, each time it passes through the vertical edge going from left to right through the right edge add $+1$ to the count and each time it passes through the vertical edge going from right to left through the left edge add -1 to the count. In the below example, the green curve has intersection numbers (vertical,horizontal) given by $(2, 2)$.



(a) Give two examples of curves with each of the following (v,h) intersection numbers $(1, 1)$, $(1, 0)$, $(1, 3)$, $(2, 3)$.

(b) Give an example of two $(1, 0)$ curves which are homotopic (give the homotopy and make sure it is well-defined in the torus given the identifications of opposite edges). Do the same with two $(1, 1)$ curves.

(c) See if you can give a proof that any two $(1, 0)$ curves are homotopic. Then if you can, try to prove that if two curves have the same (v,e) numbers then they are homotopic in the torus.