MORSE THEORY

Reading: Matsumoto, *Introduction to Morse Theory*

1. Morse theory for 1-dimensional manifolds

Here is a basic model of a function which has derivative zero at a point: $f(x) = x^2$.

This is a function from $\mathbb{R}$ to $\mathbb{R}$ and the derivative is $\frac{df}{dx}(x) = 2x$. When $x = 0$, the derivative is zero: $\frac{df}{dx}(0) = 0$.

When all the partial derivatives of a function are zero at a point, we say that point is a *critical point* of the function. So in this case $0 \in \mathbb{R}$ is a critical point of the function $f(x) = x^2$. The function value at a critical point is called a *critical value*. In this case $f(0) = 0$ so 0 is a critical value.

Question: Compare the preimage of values just above and just below the critical value here: what is $f^{-1}(-\varepsilon)$ and $f^{-1}(\varepsilon)$ where ε is a small positive number. What is $f^{-1}(0)$? How do these preimages fit together to give the graph of f?

Question: Work out the same ideas and questions for $g(x) = -x^2$.

Question: Suppose $h : \mathbb{R} \to \mathbb{R}$ is a function and there are no critical values in the interval $[c, d]$. Show that the subset of the graph between height c and d,

$$G(c, d) = \{(x, h(x)) \mid c \leq h(x) \leq d\}$$

is homeomorphic to the product $h^{-1}(c) \times [c, d]$.

You now have your basic building blocks for 1-dimensional manifolds: the graph of x^2 near the origin, the graph of $-x^2$ near the origin, and continuing upwards with intervals. Each of these building blocks gives you an open piece of a manifold. In fact we can understand most functions $h : M \to \mathbb{R}$ where M is a 1-dimensional manifold in this way.

Suppose we have a 1-dimensional manifold M and a continuous smooth function $h : M \to \mathbb{R}$. Since M is a 1-dimensional manifold, around every point, we can find an open set $U \subset M$ such that U is homeomorphic to an open subset $V \subset \mathbb{R}$. Let $\phi : V \to U$ be the homeomorphism. Then $h \circ \phi : V \to \mathbb{R}$ is a function from a subset of $\mathbb{R}$ to $\mathbb{R}$. Suppose that for any critical point $p \in M$ of the function h, there is a homeomorphism ϕ such that the composition $h \circ \phi(x) = \pm x^2$. This turns out to be possible for most functions h.

In figure 1 is an example of a graph of a function from a 1-dimensional manifold to $\mathbb{R}$ with two critical points of type x^2 and 2 critical points of type $-x^2$.

Question: Draw two examples of graphs of 1-dimensional manifolds with 3 critical points of the type x^2 and 3 critical points of the type $-x^2$.

Question: Give examples of graphs of 1-dimensional manifolds with n critical points of type x^2 and m critical points of type $-x^2$ for as many pairs (n, m) as you can. Are there restrictions on the possibilities for (n, m)?

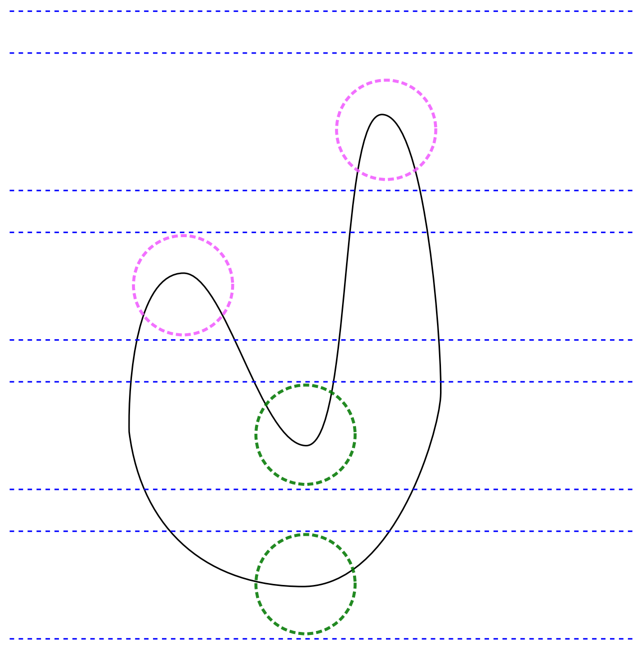

FIGURE 1. A graph of a function from a 1-dimensional manifold to $\mathbb{R}$ with two critical points modeled on x^2 and two critical points modeled on $-x^2$.

2. Morse theory for 2-dimensional manifolds

The models for critical points of a map from a 2-dimensional manifold to $\mathbb{R}$ are the following:

$$f_0(x,y) = x^2 + y^2$$
$$f_1^+(x,y) = x^2 - y^2$$
$$f_1^-(x,y) = -x^2 + y^2$$
$$f_2(x,y) = -x^2 - y^2$$

Each function has both partial derivatives equal to zero at the origin $(0,0)$, which means $(0,0)$ is a critical point for each of these functions. The critical value in each case is also 0. For each of these, find $f^{-1}(\varepsilon)$ and $f^{-1}(-\varepsilon)$ and sketch how the graph of f gives a manifold connecting $f^{-1}(-\varepsilon)$ to $f^{-1}(\varepsilon)$.

Problem: Put together these models to build a 2 sphere using one f_0 piece and one f_2 piece. Find another way to build a 2-sphere using one f_0, one f_1 and two f_2 pieces. Build each orientable surface by putting some combination of these building blocks together. What restrictions do you have on the number and types of pieces you need to use to build each surface?