PROJECTIVE SPACES

The *real n-dimensional projective space*, $\mathbb{R}P^n$ is the quotient of $\mathbb{R}^{n+1} \setminus 0$ by the equivalence relation $(x_1, \cdots, x_{n+1}) \sim (\lambda x_1, \cdots, \lambda x_{n+1})$ for $\lambda \in \mathbb{R} \setminus 0$. A typical way to represent $\mathbb{R}P^n$ is to use *homogeneous coordinates* as follows:

$$\mathbb{R}P^n = \{[x_1 : x_2 : \cdots : x_n, x_{n+1}] \mid (x_1, x_2, \cdots, x_n, x_{n+1}) \in \mathbb{R}^{n+1} \setminus (0, 0, \cdots, 0, 0)\}$$

where $[x_1 : x_2 : \cdots : x_n : x_{n+1}] = [\lambda x_1 : \lambda x_2 : \cdots : \lambda x_n : \lambda x_{n+1}]$.

(1) Explain how $\mathbb{R}P^n$ can be thought of as the space of lines in $\mathbb{R}^{n+1}$ that pass through the origin.

Let $X_1, X_2 \subset \mathbb{R}P^n$ be the subsets

$$X_1 = \{[x_1 : \cdots : x_{n+1}] \mid x_1 = 1\}$$
$$Y_1 = \{[x_1 : \cdots : x_{n+1}] \mid x_1 = 0\}$$

(2) Show that $\mathbb{R}P^n = X_1 \cup Y_1$.
(3) Show that X_1 is homeomorphic to $\mathbb{R}^n$.
(4) Show that Y_1 is homeomorphic to $\mathbb{R}P^{n-1}$.

Inductively define $X_k, Y_k \subset Y_{k-1}$ for $2 \leq k \leq n+1$ by

$$X_2 = \{[x_1 : \cdots : x_{n+1}] \mid x_1 = \cdots = x_{k-1} = 0, x_k = 1\}$$
$$Y_2 = \{[x_1 : \cdots : x_{n+1}] \mid x_1 = \cdots = x_{k-1} = x_k = 0\}$$

(5) Show that $\mathbb{R}P^n = X_1 \cup X_2 \cup \cdots \cup X_n \cup X_{n+1}$.
(6) Show that X_k is homeomorphic to $\mathbb{R}^{n-k+1}$.
(7) Conclude that $X_1 \cup X_2 \cup \cdots \cup X_n \cup X_{n+1}$ gives a cell decomposition of $\mathbb{R}P^n$. How many k-cells are there for each k?
(8) Explain how to get the polygonal presentation for $\mathbb{R}P^2$ from this cell decomposition.

Analogously, using the same equations, just changing real numbers to complex numbers, we can define the *complex n-dimensional projective space* $\mathbb{C}P^n$ as the quotient of $\mathbb{C}^{n+1} \setminus 0$ by the equivalence relation $(z_1, \cdots, z_{n+1}) \sim (\lambda z_1, \cdots, \lambda z_{n+1})$ for $\lambda \in \mathbb{C} \setminus 0$. $\mathbb{C}P^n$ also has homogeneous coordinates:

$$\mathbb{C}P^n = \{[z_1 : z_2 : \cdots : z_n, z_{n+1}] \mid (z_1, z_2, \cdots, z_n, z_{n+1}) \in \mathbb{C}^{n+1} \setminus (0, 0, \cdots, 0, 0)\}$$

where $[z_1 : z_2 : \cdots : z_n : z_{n+1}] = [\lambda z_1 : \lambda z_2 : \cdots : \lambda z_n : \lambda z_{n+1}]$.

(9) Use the analogous decomposition of $\mathbb{C}P^n$ into $X_1 \cup \cdots \cup X_{n+1}$. What is the dimension (over the reals) of each cell?