MAT 145: Homework 2 Solution

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1. (a) We abbreviate those sets by N, C and S. Then $|N \cup C \cup S| = |N| + |C| + |S| - |N \cap C| - |C \cap S| - |N \cap S| + |N \cap C \cap S| = 440$.

   (b) We have

$$
\begin{aligned}
|(N \cap C) \cup (C \cap S) \cup (N \cap S)| &= |N \cap C| + |C \cap S| + |C \cap S| \\
&\quad - |(N \cap C) \cap (C \cap S)| + |(N \cap C) \cap (N \cap S)| + |(C \cap S) \cap (N \cap S)| \\
&\quad + |(N \cap C) \cap (C \cap S) \cap (N \cap S)| \\
&= |N \cap C| + |C \cap S| + |C \cap S| - 2|(N \cap C) \cap (C \cap S) \cap (N \cap S)| \\
&= 250
\end{aligned}
$$

2. Let A be the subsets containing 3 and B be the subsets containing 7. We want to find $|A \cup B|$, which is equal to $|A| + |B| - |A \cap B|$. Note that $|A| = |B| = 2^8$ and $|A \cap B| = 2^7$, as $A \cap B$ is the subsets containing both 3 and 7. So $|A \cup B| = 2^9 - 2^7$.

3. Let A and B be the set of numbers that are multiple of 5 and 7, respectively. We want to find $|A^c \cap B^c|$, which is equal to $|(A \cup B)^c| = 998 - |A \cup B| = 998 - (|A| + |B| - |A \cap B|)$. Note that $|A| = 199$, $|B| = 142$ and $|A \cup B| = 28$. So, $|A^c \cap B^c| = 998 - (199 + 142 - 28) = 685$.

4. Let X, Y and Z be the set of students that are in three teams.

   Case 1. If there is at least one student, say Alice, taking exactly one team, i.e., Alice $\in X$, but Alice $\notin Y$ and Alice $\notin Z$. Then it follows that every one in this class should be in X in order to fulfill the condition that "for any two students in the class, there is at least one team such that those two students are both members of that team". So the result follows easily.

   Case 2. Assume that every one is in at least two teams. Let $W = X \cap Y \cap Z$ be the students who are in all three teams and $w = |W|$. Let $X' = X \setminus W$ be the students that are in X but are not in all three teams. Define Y' and Z' similarly. Clearly, we have $|X| = |X'| + w$ and similarly for Y and Z. Then $n - w = |X'| + |Y'| + |Z'| - (|X' \cup Y'| + |Y' \cup Z'| + |X' \cup Z'|) =$

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$|Z'| + |Y'| + |Z'| - (n - w)$. So, $|X'| + |Y'| + |Z'| = 2(n - w)$. So, $|X| + |Y| + |Z| = 2n + w$. So, the result follows.

5. Let A and B be the set of numbers that are perfect squares and perfect cubes, respectively. We want to find $|A \cap B|$, which is equal to $|A| + |B| - |A \cap B|$. Note that $|A| = 1000$, $|B| = 100$ and $|A \cup B| = 10$, since a number is a perfect squares and perfect cubes is equivalent to that it is a perfect power of 6. So, $|A \cap B| = 1000 + 100 - 10 = 1090$.

6. (a) Let the value be x. Then $70 - 7 = 15 + 25 + 50 - x + 1$, implying that $x = 28$.
   (b) Similar to 1(b), we have $28 - 2 \times 1 = 26$.
   (c) There are 26 students taking at least two (i.e., two or three) classes, by part (b). One of them is taking all three classes. So the answer is $26 - 1 = 25$.

7. Fix L_0. Assume, for contradiction, that there are at most 4 points on L_0. For any one of those 4 points, by condition, at most two other lines pass through it. So, there are at most 4×2 lines intersecting with L_0. But all other 9 lines should intersect with L_0 since no two of the 10 lines are parallel. Contradiction.

8. (a) Sort the numbers such that $x_1 < x_2 < \cdots < x_{n+1}$. Then $x_2 - x_1 \geq 1, x_3 - x_2 \geq 1, \ldots, x_{n+1} - x_n \geq 1$. Summing up these n inequalities, we have $x_{n+1} - x_1 \geq n$.
   (b) For any $x \in X$, we may write $x = an + r_x$, such that the quotient r_x satisfies $0 \leq r_x < n$, taking at most n possible values. So, there have to be x and y such that $r_x = r_y$ by pigeon hole principle. So, $x - y$ is divisible by n.

9. Partition the dartboard into 64 1 by 1 squares. Then at least one square takes at least two darts. The distance between these two is less than $\sqrt{2} < 1.5$.