MAT 145: Homework 3 Solution

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1. (a) Choosing k people to form the group gives $\binom{n}{k}$ then choosing 1 person from those k gives the desired result.

   (b) Consider completing the following task: Choose a subset of n people, which contains exactly one person as the president. Using two methods to count how many ways to complete this task should lead to the same result. One method is to form a k-people council, for $k = 1, \ldots, n$, giving the left hand side. Another method is to select a president first (giving n ways), then select a subset from the remaining $n-1$ people to form the council. This gives the right hand side.

2. (a) Consider an n-element set X. To describe in set language, the task is to choose a k-element subset A and then choose an s-element subset $B \subset A$. Clearly, $\binom{n}{k}\binom{k}{s}$.

   (b) Using two methods to count how many ways to complete this task should lead to the same result. One method is given in (a), which gives the left hand side. Another method is that we choose the element in B first, $\binom{n}{s}$ ways. Then fill in A by choosing $k-s$ elements from the remaining $n-s$ elements.

3. Plugging in $x = 1$ and $y = 2$ in the binomial expansion $(x+y)^n$.

4. This identity follows from the identity in problem 5, by plugging in m and k by n and noting that $\binom{n}{\ell} = \binom{n}{n-\ell}$.

5. Consider completing the following task: Choose k balls from two bags, where the first contains n different red balls and the second one contains m different blue balls. Using two methods to count how many ways to complete this task should lead to the same result. One method is to select ℓ red balls and $k-\ell$ blue balls, for $\ell = 0, \ldots, k$. This gives that left hand side. Another method is to select k balls from two bags at once, which gives the right hand side.

6. In 11 positions, choose 2 to put M: $\binom{11}{2}$. Choose 2 to put A from the remaining 9: $\binom{9}{2}$. ... So, $\binom{11}{2}\binom{9}{2}\binom{7}{2}\binom{5}{1}\binom{4}{1}\binom{3}{1}\binom{2}{1}\binom{1}{1}$.

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7. Equivalently to the above problem, there are $\binom{10}{4}\binom{6}{2}\binom{4}{2}\binom{2}{1}\binom{1}{1}$ ways to distribute. Those ways happen with same probability. So the probability that the waiter delivers the right dish to everyone is $1/\left[\binom{10}{4}\binom{6}{2}\binom{4}{2}\binom{2}{1}\binom{1}{1}\right]$.

8. (a) There are $\binom{49}{6}$ ways.
   (b) Choose a 6-element multiset from 49 elements. There are $\binom{49+6-1}{6}$ ways.

9. (a) $\dfrac{\binom{50}{5}\binom{50}{5}}{\binom{100}{10}}$.
   (b) $\dfrac{\binom{2n}{n}\binom{2n}{n}}{\binom{4n}{2n}}$.
   See any connection between this question and question 4 or 5?