MAT 145: Homework 4 Solution

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1. (a) To achieve $(13, 9)$, one needs to make 22 moves, among which 13 are *right* moves and 9 are *up* moves. As a result, there are $\binom{22}{13}$ such paths.

   (b) Similarly, there are $\binom{22}{13}$ such paths.

   (c) One needs to make 10 moves, among which 4 are *right* moves and 6 are *up* moves. So, $\binom{10}{4}$.

   (d) There are $\binom{20}{8}$ moves from $(0, 0)$ to $(8, 12)$. We first count the number of paths that pass through $(4, 2)$. There are $\binom{4+2}{4}\binom{(8-4)+(12-2)}{8-4} = \binom{6}{4}\binom{14}{4}$ such paths. So, there are $\binom{20}{8} - \binom{6}{4}\binom{14}{4}$ paths that do not pass through $(4, 2)$.

2. We fix an arbitrary n and prove by induction on k. If $k = 0$, then $\binom{n}{0} = \binom{n+1}{0}$.

   Now assume that

$$\binom{n}{0} + \binom{n+1}{1} + \cdots + \binom{n+k-1}{k-1} = \binom{n+k-1+1}{k-1}.$$

   Then

$$\begin{aligned} \binom{n}{0} + \binom{n+1}{1} + \cdots + \binom{n+k-1}{k-1} + \binom{n+k}{k} &= \binom{n+k-1+1}{k-1} + \binom{n+k}{k} \\ &= \binom{n+k+1}{k} \end{aligned}$$

3. See `https://oeis.org/wiki/Template:Sierpinski%27s_triangle_(Pascal%27s_triangle_mod_2)`.

   This binary figure (known as Sierpinski triangle) is closely related to a kind of discrete dynamical system, cellular automata. Such self-similar phenomenon in cellular automata is very common and known as fractal or replication. Talk to your TA if your are interested.

4. An equivalent question is how many integer solutions does the equation

$$x_1 + x_2 + x_3 + x_4 = 50$$

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have, such that $x_j \geq 5$, for $j = 1, 2, 3, 4$. Let $y_j = x_j - 5$. Then this is equivalent to asking the number of non-negative integer solutions of the equation

$$y_1 + y_2 + y_3 + y_4 = 30.$$

The answer is $\binom{30+4-1}{4-1} = \binom{33}{3}$.

5. $\binom{100}{8}\binom{92}{9}\binom{83}{11}\binom{72}{5}\binom{67}{5}\binom{62}{10}$.

6. (a) Equivalently: number of positive integer solutions to the equation

$$x_1 + \cdots + x_k = n.$$

Let $y_j = x_j + 1$, then equivalently: number of non-negative integer solutions to the equation

$$y_1 + \cdots + y_k = n - k,$$

which is

$$\binom{n - k + k - 1}{k - 1}.$$

Note that this answer works well even when $n < k$.

(b) Equivalently: number of non-negative integer solutions to the equation

$$x_1 + \cdots + x_k = n,$$

which is

$$\binom{n + k - 1}{k - 1}.$$

(c) Equivalently: number of non-negative integer solutions to the equation

$$x_1 + \cdots + x_k = n,$$

so that $x_1, \ldots, x_\ell \geq 1$. Let $y_j = x_j + 1$, for $j = 1, \ldots, \ell$. Then equivalently: number of non-negative integer solutions to the equation

$$y_1 + \cdots + y_\ell + x_{\ell+1} + x_k = n - \ell,$$

which is

$$\binom{n - \ell + k - 1}{k - 1}.$$

7. (a) $$\binom{31 - 5 + 5 - 1}{5 - 1} = \binom{30}{4}.$$

(b) $\binom{31+5-1}{5-1} = \binom{35}{4}$.

8. Let this number be A_n. Then $A_1 = 2$: $\emptyset$ and $\{1\}$; $A_2 = 3$: $\emptyset$, $\{1\}$ and $\{2\}$.

   To find A_n, if a subset contains the number n, then it must not contain the number $n-1$, so it suffices to pick a subset containing no consecutive integers from $\{1, 2, \ldots, n-2\}$: there are A_{n-2} ways to do so; if a subset does not contain the number n, then it suffices to pick a subset containing no consecutive integers from $\{1, 2, \ldots, n-1\}$: there are A_{n-1} ways to do so. As a result, $A_n = A_{n-1} + A_{n-2}$.

   Considering the initial conditions, we have $A_n = F_{n+2}$, for $n = 1, 2, \ldots$

9. Use induction on n. For $n = 1$, we have $F_5 = 5$, which is divisible by 5. Assume that $F_{5(n-1)} \mid 5$. Then

$$\begin{aligned} F_{5n} &= F_{5n-1} + F_{5n-2} \\ &= F_{5n-2} + 2F_{5n-3} + F_{5n-4} \\ &= \ldots \\ &= 5F_{5n-4} + 2F_{5n-5}, \end{aligned}$$

   which is clearly divisible by 5.

10. Use induction on n. For $n = 1$, we have $F_1 = 1 = F_2$. Assume $F_1 + F_3 + \cdots + F_{2n-3} = F_{2n-2}$. Then $F_1 + F_3 + \cdots + F_{2n-3} + F_{2n-1} = F_{2n-2} + F_{2n-1} = F_{2n}$.

11. Use induction on n. For $n = 1$, we have $F_1^2 = 1 = 1 \times 1 = F_1 \cdot F_2$. Assume that $F_1^2 + \cdots + F_{n-1}^2 = F_{n-1} \cdot F_n$. Then $F_1^2 + \cdots + F_{n-1}^2 + F_n^2 = F_{n-1} \cdot F_n + F_n^2 = F_n(F_{n-1} + F_n) = F_n F_{n+1}$.

12. Use induction on n. For $n = 2$, we have $F_1 F_3 - F_2^2 = 2 - 1 = 1 = (-1)^2$. Assume $F_{n-2}F_n - F_{n-1}^2 = (-1)^{n-1}$. Then $F_{n-1}F_{n+1} - F_n^2 = F_{n-1}(F_{n-1} + F_n) - F_n(F_{n-2} + F_{n-1}) = -(F_{n-2}F_n - F_{n-1}^2) = (-1)^n$.