MAT 145: Homework 7 Solution

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1. The tree is

$$0 - -1 - -4 - -6 - -2 - -3$$

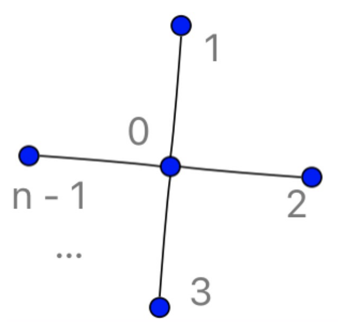
$$|$$

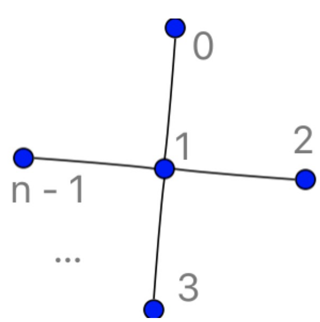
$$5$$

The two line code is

$$\begin{array}{cccccc} 3 & 5 & 2 & 6 & 4 & 1 \\ 2 & 2 & 6 & 4 & 1 & 0 \end{array}$$

2.





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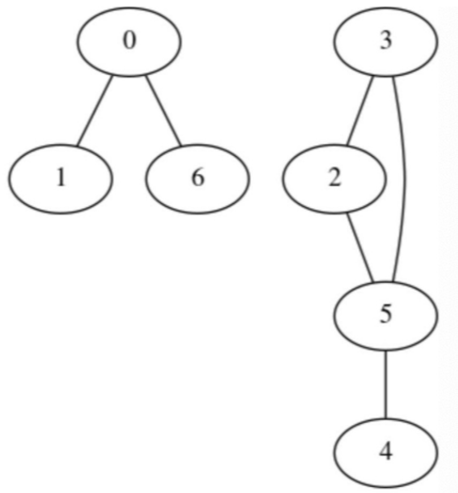
3. The tree is uniquely determined by the code 03552. It gives the tree

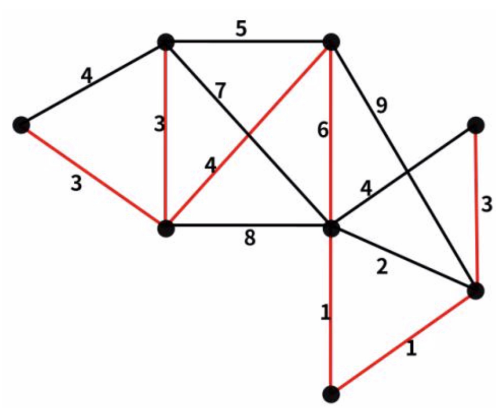
$$\begin{array}{c} 1 - -0 - -2 - -5 - -3 - -4 \\ | \\ 6, \end{array}$$

which does not give the two-line code.

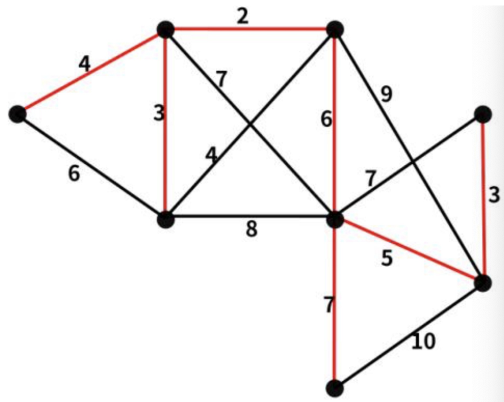
The graph corresponding to the two-line code is



4. (a)



(b)



5. (a) The matrix is

$$A = \begin{bmatrix} 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

(b)

$$A^2 = \begin{bmatrix} 3 & 0 & 2 & 0 & 0 & 1 & 1 \\ 0 & 2 & 0 & 2 & 1 & 0 & 0 \\ 2 & 0 & 2 & 0 & 0 & 1 & 0 \\ 0 & 2 & 0 & 3 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 2 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 2 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 2 \end{bmatrix}$$

The verification is direct. For example, let $A^k = \left(a_{ij}^{(k)}\right)_{i,j=1,\ldots,n}$. Then $a_{1,3}^{(2)} = 2$ means there are two length-2 paths from 1 to 3. Indeed, 143 and 123 are those two paths.

(c) By the definition of matrices multiplication, $a_{ij}^{(2)} = \sum_{\ell=1}^{n} a_{i\ell}a_{\ell j}$. Note $a_{i\ell}a_{\ell j} = 1$ if and only if $a_{i\ell} = a_{\ell j} = 1$ if and only if there is a path from i to ℓ to j. So, $\sum_{\ell=1}^{n} a_{i\ell}a_{\ell j}$ counts the number of paths from i to j, passing from another vertex.

(d) We use induction on k. The base case $k = 1$ is the definition of A. Assume that A^{k-1}

gives the number of length k walks from vertex i to vertex j. Then

$$A^k = \sum_{\ell=1}^{n} a_{i\ell}^{(k-1)} a_{\ell j}.$$

Note that $a_{i\ell}^{(k-1)} a_{\ell j}$ counts the number of length-k paths from i to j, passing ℓ as the second last vertex. So, $\sum_{\ell=1}^{n} a_{i\ell}^{(k-1)} a_{\ell j}$ counts all length-k paths from i to j.

The corollary is straightforward: The ith row and jth column of $A + A^2 + \ldots, A^{n-1}$ is positive if and only if $a_{ij}^{(k)} > 0$, for some $k = 1, 2, \ldots, n-1$ if and only if there is a length-k path from i to j for some k. The graph is connected if and only if the above is true for all i and j.

6. We use the induction on k. When $k = 1$, G is a tree and the result follows from a theorem. Assume the result is true when G has n vertices and $k-1$ components, i.e., in this case, G has $n - (k-1)$ edges.

   Now, if G has k components, add one edge to G to connect any two components. Call this new graph G'. G' has n vertices and $k-1$ components, thus has $n - (k-1)$ edges by hypothesis. Deleting the new edges recovers G, which has $n - (k-1) - 1 = n - k$ edges.

7. The graph must have two connected components. Otherwise, if it has more than two components, adding one edge that connects two component will increase the number of edges and keep the graph disconnected.

   So, let one component contain m vertices and the other contains $n-m$ vertices. To maximize the number of edges, make the two component contain the maximal possible number of edges, respectively, i.e., make them K_m and K_{n-m}. So, the total number of vertices is

$$\binom{m}{2} + \binom{n-m}{2} = m^2 - 2nm + (n^2 - n)/2,$$

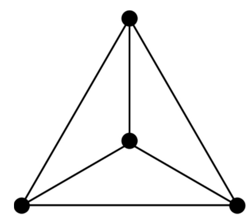
   which is a quadratic function in terms of m for $m = 1, 2, \ldots, n-1$. It attains maximum when $m = 1$ or $n - 1$. In either case, the number of edges is

$$\binom{n-1}{2} = \frac{(n-1)(n-2)}{2}.$$

8. (a) This is $K_{3,3}$, so it has no planar embedding.

   (b) A possible planar embedding of K_4:

   (c) The complete graph K_6 has the graph in (a), i.e., $K_{3,3}$, as a subgraph. So, K_6 has no planar embedding.



(d) Every tree has a planar embedding. We can construct the trees by adding one new vertex connected by one new edge to an existing vertex. We can draw this new edge without crossing any other edge if we draw the new vertex very close to the original one so that no other edge can get in between. Since we can do this at every stage of the construction, we can draw the entire tree in the plane without any edges crossing.

9. (a) If T is a tree, e is an edge in the tree with endpoints v_0 and v_1, and T' is the contraction of T along e, we can verify T' is connected as in class because the contraction of a linear subgraph is a linear subgraph. Next, we show T' has no cycles. Suppose for contradiction that T' had a cycle. Then un-contracting the edge e to get back to T, the cycle in T' will either remain in T or will grow to a cycle one edge longer in T. In either case, this will imply that T has a cycle but T is a tree so we get a contradiction.

(b) Use induction on n. When $n = 1$, the graph is a single vertex. Thus the result follows. Assume that a $(n-1)$-vertex tree has $n-2$ edges. Now consider a tree T with n vertices. Contract an edge to obtain T'. Now, T' has $n-1$ vertices thus has $n-2$ edges. Note that contraction reduces the number of vertices and number of edges by 1, respectively. So, T has $n - 2 + 1 = n - 1$ edges.