Final Exam

Math 145, Spring 2019

Name: Solutions

Student ID: ____________________

Every solution must contain an explanation written in words supporting your numerical solution to receive credit.

You do not need to simplify numerical expressions for your final answers (e.g. you can write $2^8 \cdot 4!$ instead of multiplying out to 6144.)

If you need extra space for your solutions, there is an extra page at the back of the exam. If you need extra space for any problem, write CONTINUED IN EXTRA SPACE on the page where the problem is given to you. In the extra space write the problem number that you are solving in that space.

Problem 1: Let T be a tree. Prove that every edge of T is a cut-edge.

Let e be any edge of T. Let v_i and v_j be the endpoints of e.

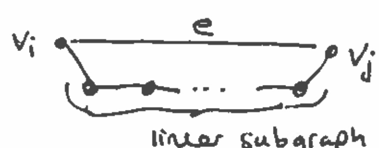
Let T' be the graph obtained from T by deleting e, from the edge set.

Suppose for contradiction that T' was connected.

Then there exists a linear subgraph of T' connecting v_i to v_j

Since e is not an edge of T', e is not included in this linear subgraph.

Then the union of the linear subgraph with the edge e is a subgraph of T, but this subgraph is a cycle



but T is a tree so it cannot contain a cycle.

Problem 2: How many numbers between 1 and 100 are NOT a multiple of any of the numbers 3, 5, 11?

Define $A, B, C \subseteq \{1, 2, \dots, 100\}$ by: (arithmetic mistakes were forgiven)

$A = \{\text{multiples of } 3\}$ $\quad |A| = \lfloor 100/3 \rfloor = 33$

$B = \{\text{multiples of } 5\}$ $\quad |B| = \lfloor 100/5 \rfloor = 20$

$C = \{\text{multiples of } 11\}$ $\quad |C| = \lfloor 100/11 \rfloor = 9$

$A \cap B = \{\text{multiples of } 3 \cdot 5 \cdot 15\}$ $\quad |A \cap B| = \lfloor 100/15 \rfloor = 6$

$A \cap C = \{\text{multiples of } 3 \cdot 11 = 33\}$ $\quad |A \cap C| = \lfloor 100/33 \rfloor = 3$

$B \cap C = \{\text{multiples of } 5 \cdot 11 = 55\}$ $\quad |B \cap C| = \lfloor 100/55 \rfloor = 1$

$A \cap B \cap C = \{\text{multiples of } 3 \cdot 5 \cdot 11 = 165\} = \emptyset$ $\quad |A \cap B \cap C| = 0$

Then numbers which are a multiple of some number 3, 5, 11 are

Inclusion-Exclusion Principle

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

$$= 33 + 20 + 9 - 6 - 3 - 1 + 0$$

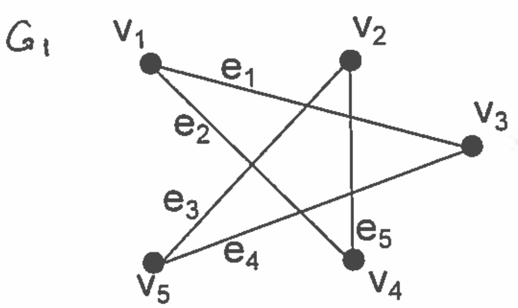
$$= 52$$

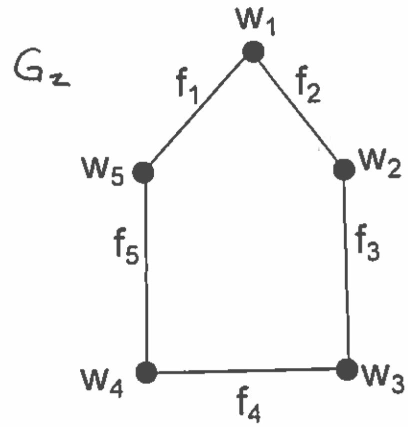
So the numbers which are not a multiple of any of these are

$\{1, 2, \dots, 100\} \setminus (A \cup B \cup C)$

so there are $100 - 52 = \boxed{48}$ such numbers.

Problem 3: For the following pair of graphs, either prove they are isomorphic or prove they are not isomorphic.





We construct an isomorphism

$\Phi(v_1) = w_1$	$\Psi(e_1) = f_2$
$\Phi(v_2) = w_4$	$\Psi(e_2) = f_1$
$\Phi(v_3) = w_2$	$\Psi(e_3) = f_4$
$\Phi(v_4) = w_5$	$\Psi(e_4) = f_3$
$\Phi(v_5) = w_3$	$\Psi(e_5) = f_5$

endpoints in G_1	endpoints in G_2
e_1: v_1, v_3	$\Psi(e_1) = f_2$: $w_1, w_2 = \Phi(v_1), \Phi(v_3)$
e_2: v_1, v_4	$\Psi(e_2) = f_1$: $w_1, w_5 = \Phi(v_1), \Phi(v_4)$
e_3: v_2, v_5	$\Psi(e_3) = f_4$: $w_3, w_4 = \Phi(v_5), \Phi(v_2)$
e_4: v_3, v_5	$\Psi(e_4) = f_3$: $w_2, w_3 = \Phi(v_3), \Phi(v_5)$
e_5: v_4, v_2	$\Psi(e_5) = f_5$: $w_4, w_5 = \Phi(v_2), \Phi(v_4)$

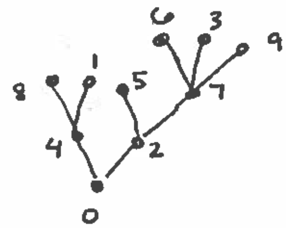
Other solutions:

$v_1 \to w_1$	$e_1 \to f_1$	w_2	f_2	w_5	f_5	w_4	f_5	w_4	f_4	w_3	f_4	w_2	f_3
$v_2 \to w_3$	$e_2 \to f_2$	w_4	f_3	w_2	f_1	w_2	f_4	w_1	f_5	w_1	f_3	w_5	f_2
$v_3 \to w_5$	$e_3 \to f_4$	w_1	f_5	w_4	f_3	w_5	f_2	w_3	f_2	w_4	f_1	w_3	f_5
$v_4 \to w_2$	$e_4 \to f_5$	w_3	f_1	w_1	f_4	w_3	f_1	w_5	f_3	w_2	f_5	w_1	f_4
$v_5 \to w_4$	$e_5 \to f_3$	w_5	f_4	w_3	f_2	w_1	f_3	w_2	f_1	w_5	f_2	w_4	f_1

Problem 4: Let T be the labeled tree that corresponds to the Prüfer code 47274072.
(a) Draw the tree T **and** *(b) write down the degree sequence for the vertices of T.*

(a)

1	3	5	6	8	4	9	7	2
4	7	2	7	4	0	7	2	0



(b) Degrees:

Vertex	0	1	2	3	4	5	6	7	8	9
Degree	2	1	3	1	3	1	1	4	1	1

Ordered: (4, 3, 3, 2, 1, 1, 1, 1, 1, 1)

Problem 5: Suppose G is a simple connected graph with a planar embedding that cuts up the plane into 8 regions. If G has 12 vertices,

(a) Calculate (with proof) the sum of the degrees of all the vertices.

$$\sum_{v \in V(G)} d(v).$$

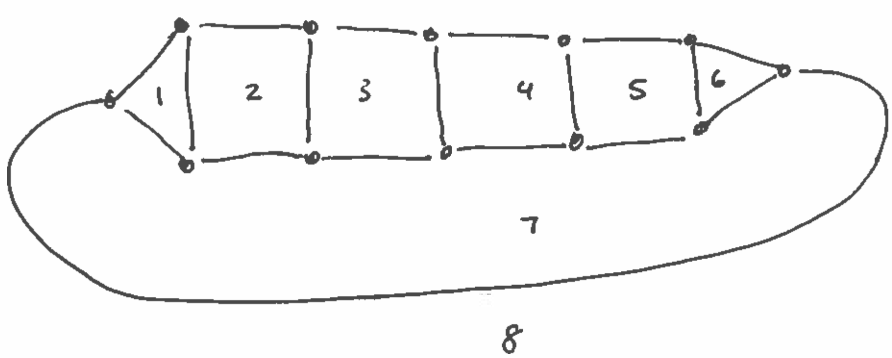
Euler's formula: $\#V - \#E + \#R = 2$

$$12 - \#E + 8 = 2$$

$$\Rightarrow \quad 18 = \#E$$

Edge-degree Formula: $\sum_{v \in V(G)} d(v) = 2\#E = \boxed{36}$

(b) Next, draw an example of a planar embedding of a simple connected graph G with these properties (8 regions in the planar embedding and 12 vertices).



Problem 6: Give the proof for the statement that if a graph G has a closed Eulerian walk, then every vertex of G has even degree.

An Eulerian walk uses every edge exactly once.

If it is a closed Eulerian walk, the starting + ending vertex are the same.

Therefore every vertex is entered on the walk the same number of times it is exited.

Since every edge must be used, every endpoint of every edge must be used either as an entering point or an exiting point.

The degree of a vertex v is the total number of endpoints of edges that agree with v.

Since half of these endpoints must be entering points and the other half must be exiting points the total number of endpoints of edges at v must be even.

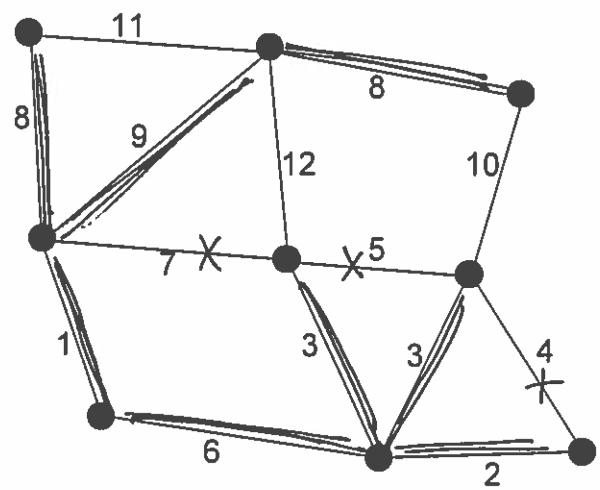
Therefore the degree at v must be even.

Problem 7: Carla is choosing the dinner buffet for a celebration. First, she must choose whether to serve 3, 4 or 5 different main course dishes. Once she decides how many dishes to serve, she must pick the dishes from a list of 12 main course options. Then she has to choose 2 dessert choices from a list of 6 dessert options. How many different possible dinner and dessert combination selections does Carla have? (We consider all dinners with 3 main course options different from all dinners with 4 main course options, etc.)

$$\left[\underbrace{\binom{12}{3}}_{\text{\# choices of 3 dish meals}} + \underbrace{\binom{12}{4}}_{\text{\# choices of 4 dish meals}} + \underbrace{\binom{12}{5}}_{\text{\# choices of 5 dish meals}}\right] \cdot \underbrace{\binom{6}{2}}_{\text{\# choices of dessert}}$$

Given n options, we want to choose k of them, unordered, non repeating so the number of such options is $\binom{n}{k}$.

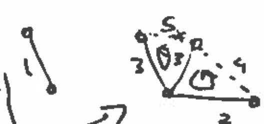
Problem 8: Find the minimal cost spanning tree for the following weighted graph. Draw the spanning tree and determine its total cost.



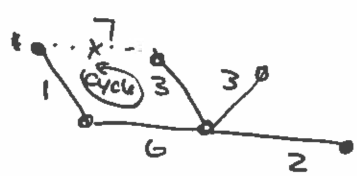
1 Include
2 Include
3 Include
3 Include

no cycles

4 exclude because would create a cycle
5 exclude b/c would create a cycle

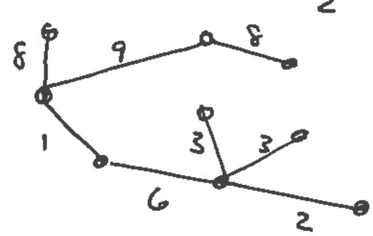


6 Include



7 exclude - cycle

8 Include
8 Include
9 Include



Now all vertices are connected so we stop adding edges

$\boxed{\text{Total cost: } 1+2+3+3+6+8+8+9 = 40}$

Problem 9: Suppose G is a simple graph with 8 vertices and 17 edges. Prove that G has at least one vertex of degree strictly greater than 4.

Degree - edge formula:

$$\sum_{v \in V(G)} d(v) = 2\#E(G) = 2 \cdot 17 = 34$$

8 vertices

$$d(v_1) + d(v_2) + \dots + d(v_8) = 34$$

If every degree was less than or equal to 4 then

$$d(v_1) + d(v_2) + \dots + d(v_8) \leq \underbrace{4+4+\dots+4}_{8 \text{ times}} = 4 \cdot 8 = 32$$

But 34 is not ≤ 32

So we get a contradiction.

Thus there must be at least one vertex with degree greater than 4.

Extra Space: