Practice Final Exam 1

Math 145, Spring 2019

Name: Solutions

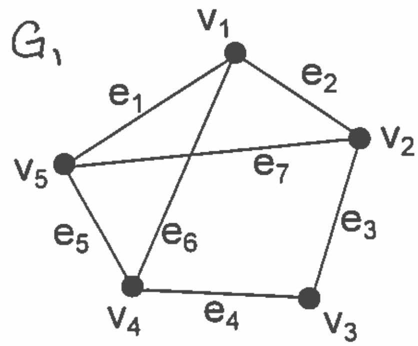
Student ID: ____________________

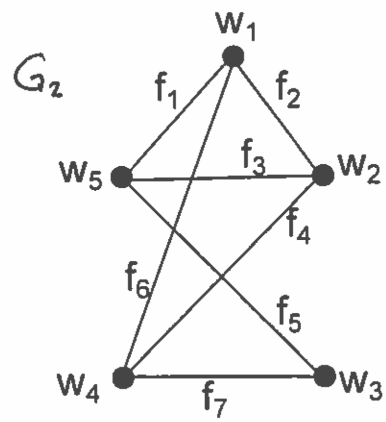
Every solution must contain an explanation written in words supporting your numerical solution to receive credit.

You do not need to simplify numerical expressions for your final answers (e.g. you can write $2^8 \cdot 4!$ instead of multiplying out to 6144.)

If you need extra space for your solutions, there is an extra page at the back of the exam. If you need extra space for any problem, write CONTINUED IN EXTRA SPACE on the page where the problem is given to you. In the extra space write the problem number that you are solving in that space.

Problem 1: For the following pair of graphs, either prove they are isomorphic or prove they are not isomorphic.





Isomorphic

Bijections:

$\Phi: V(G_1) \to V(G_2)$

$\Phi(v_1) = w_1$

$\Phi(v_2) = w_5$

$\Phi(v_3) = w_3$

$\Phi(v_4) = w_4$

$\Phi(v_5) = w_2$

$\Psi: E(G_1) \to E(G_2)$

$\Psi(e_1) = f_2$ endpts(e_1): $v_1, v_5 \xrightarrow{\Phi} w_1, w_2 =$ endpts(f_2)

$\Psi(e_2) = f_1$ endpts(e_2): $v_1, v_2 \xrightarrow{\Phi} w_1, w_5 =$ endpts(f_1)

$\Psi(e_3) = f_5$ endpts(e_3): $v_2, v_3 \xrightarrow{\Phi} w_5, w_3 =$ endpts(f_5)

$\Psi(e_4) = f_7$ endpts(e_4): $v_3, v_4 \xrightarrow{\Phi} w_3, w_4 =$ endpts(f_7)

$\Psi(e_5) = f_4$ endpts(e_5): $v_4, v_5 \xrightarrow{\Phi} w_4, w_2 =$ endpts(f_4)

$\Psi(e_6) = f_6$ endpts(e_6): $v_1, v_4 \xrightarrow{\Phi} w_1, w_4 =$ endpts(f_6)

$\Psi(e_7) = f_3$ endpts(e_7): $v_2, v_5 \xrightarrow{\Phi} w_5, w_2 =$ endpts(f_3)

Problem 2: At a store opening, they were giving out a free pencil to every 5^{th} customer who walked in (the 5^{th}, 10^{th}, 15^{th}, etc.), a free hat to every 12^{th} customer who walked in (the $12^{th}, 24^{th}, \ldots$), and a free T-shirt to every 23^{rd} customer who walked in ($23^{rd}, 46^{th}, \ldots$). If there were 500 customers, how many got a free item?

A = {people who got a pencil} $|A| = \lfloor 500/5 \rfloor = 100$

B = {people who got a hat} $|B| = \lfloor 500/12 \rfloor = 41$

C = {people who got a T-shirt} $|C| = \lfloor 500/23 \rfloor = 21$

$12 \cdot 40 = 480$
$12 \cdot 41 = 492$

$23 \cdot 20 = 460$
$23 \cdot 21 = 483$

Inclusion-Exclusion Principle

$\Rightarrow |A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$

$A \cap B$ = {people who got a pencil + hat} - must be multiple of 5 and 12 $\rightsquigarrow$ multiple of 60

$|A \cap B| = \lfloor 500/60 \rfloor = 8$

$A \cap C$ = {people who got a pencil + Tshirt} - must be multiple of 5 and 23 $\rightsquigarrow$ multiple of 115

$|A \cap C| = \lfloor 500/115 \rfloor = 4$

$B \cap C$ = {people who got a hat + Tshirt} - must be multiple of 12 and 23 $\rightsquigarrow$ multiple of 276

$|B \cap C| = \lfloor 500/276 \rfloor = 1$

$A \cap B \cap C$ = {people who got pencil + a hat + T-shirt} must be multiple of 5 + 12 + 23 $\rightsquigarrow$ multiple of $276 \cdot 5 > 500$

$|A \cap B \cap C| = \left\lfloor \frac{500}{276 \cdot 5} \right\rfloor = 0$

So by inclusion exclusion the number of people who got a free item is:

$\boxed{100 + 41 + 21 - 8 - 4 - 1 + 0}$

Problem 3: Give the proof for the statement (directly from the definitions) that if G is a graph, and $d(v)$ denotes the degree of a vertex then

$$\sum_{v \in V(G)} d(v) = 2\#E(G).$$

We prove this combinatorially

Each edge has two endpoints.

Each endpoint corresponds to one vertex $v \in V(G)$.

The degree $d(v)$ is the number of endpoints of an edge such that the endpoint corresponds to v.

Therefore :

$$\sum_{v \in V(G)} \text{number of endpoints corresponding to } v = \text{total number of end points} = \sum_{e \in E(G)} \text{number of endpoints of } e$$

$$\sum_{v \in V(G)} \text{number of endpoints corresponding to } v = \sum_{v \in V(G)} d(v)$$

$$\sum_{e \in E(G)} \text{number of endpoints of } e = \sum_{e \in E(G)} 2 = 2\#E(G)$$

$$\Rightarrow \sum_{v \in V(G)} d(v) = 2\#E(G)$$

Problem 4: Suppose G is a connected graph, and e an edge of G. Prove that e is NOT a cut-edge *if and only if* it is contained in a cycle of G.

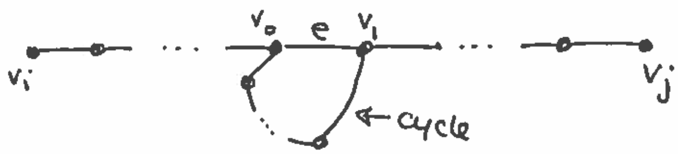
First we show if e is in a cycle of G then e is not a cut edge:

Let G' be the graph obtained by deleting e from G. We show G' is connected so e is not a cut-edge:

Let v_i and v_j be vertices of G' (thus vertices of G).

Since G is connected there is a linear subgraph of G connecting v_i to v_j. If e is not an edge of the linear subgraph, we can connect v_i to v_j by a linear subgraph of G'.

If e is an edge of the linear subgraph of G,

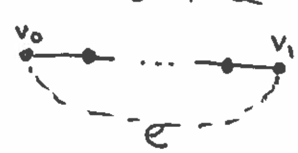


Consider the union of the linear subgraph with a cycle that e is in. Deleting e from the cycle gives a linear subgraph connecting the endpoints v_0 + v_1 of e.

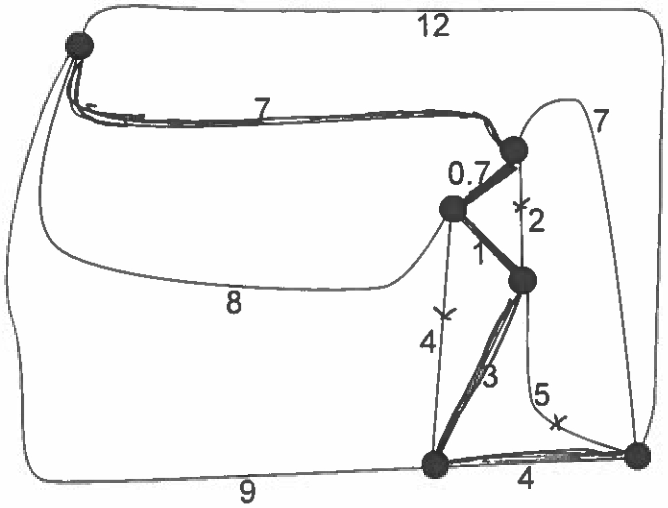
Let v be the vertex in the cycle closest to v_1 (distance measured by edges in the linear graph obtained by deleting e from the cycle) that occurs in the original linear subgraph from v_i to v_0. We know v exists because v_0 is in both the linear subgraph v_i to v_0 and in the cycle. Then we get a linear subgraph from v_i to v_j by concatenating the linear subgraph from v_i to v with the part of the cycle from v to v_1 with the linear subgraph from v_1 to v_j.

Next, if e is not a cut edge we show e is contained in a cycle.

If e is not a cut edge, let v_0 + v_1 be its endpoints and G' the graph obtained from G by deleting e. G' is connected so there is a linear subgraph of G' connecting v_0 to v_1 then adding e to this linear subgraph gives a cycle in G containing e.



Problem 5: Find the minimal cost spanning tree for the following weighted graph. Draw the spanning tree and determine its total cost.



The edges ordered by weight :

include 0.7

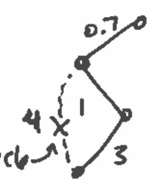
no cycle

include 1

no cycle

include 2 excluded because would form a cycle

include 3



exclude one 4 b/c creates cycle

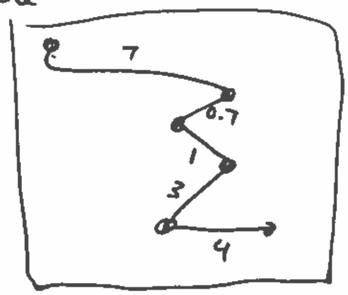
include other 4

no cycle

exclude 5 b/c creates a cycle

exclude one 7 b/c creates a cycle

include other 7 no cycle



Now done because we connected all vertices so adding any other edge would create a cycle.

Total cost:

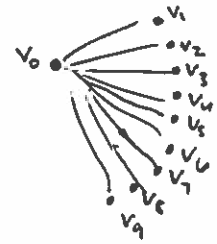
$0.7 + 1 + 3 + 4 + 7$

$= \boxed{15.7}$

Problem 6: Suppose G is a simple graph with 10 vertices and 28 edges. Prove that there are at least two vertices v_1 and v_2 such that $d(v_1) + d(v_2) \geq 12$. Next prove there are at least two other vertices v_3 and v_4 such that there is are edges connecting v_1 to v_3, v_1 to v_4, v_2 to v_3 and v_2 to v_4. Conclude G contains a cycle of length 4.

Because G is simple, there are no self-edges or multiple edges.

Since there are 10 vertices, the maximum degree of each vertex is 9



because otherwise there would be more than one edge from v_0 to v_i for some i giving a multiple edge.

By the degree-edge formula $\sum_{v \in V(G)} d(v) = 2\#E(G) = 2(28) = 56$

~~[illegible]~~

~~By the pigeonhole principle, there exist two vertices v_1, v_2 [illegible]~~

~~among the remaining 8 vertices, there must be one~~

See next page after Problem 7.
for continued solution

Problem 7: Suppose G is a graph and v_1 and v_2 are two vertices in G. Suppose there exists a walk starting at v_1 and ending at v_2. Prove that G has a linear subgraph where v_1 and v_2 are the endpoints.

If there is a walk in G starting at v_1 and ending at v_2: $W = \{v_1 = v_{i_0} e_{i_1} v_{i_1} e_{i_2} v_{i_2} \cdots v_{i_{n-1}} e_{i_n} v_{i_n} = v_2\}$

the edges cannot repeat but the vertices might repeat.

We construct a linear subgraph from v_1 to v_2 as follows:

Consider the ordered list of vertices in the walk

$$v_{i_0}, v_{i_1}, v_{i_2}, \ldots, v_{i_{n-1}}, v_{i_n}$$

Let v_{i_k} ~~[illegible]~~ and v_{i_ℓ} be the first time we have a repeated vertex $v_{i_k} = v_{i_\ell}$.

Replace the walk W with a new walk deleting the part between v_{i_k} and v_{i_ℓ}

$$W_1 = \{v_1 = v_{i_0} e_{i_1} \cdots e_{i_k} v_{i_k} = v_{i_\ell} e_{i_{\ell+1}} v_{i_{\ell+1}} \cdots e_{i_n} v_{i_n}\}$$

Repeat this process until there are no more repeated vertices. This will terminate after finitely many repetitions because there are finitely many edges in the walk and each modification of the walk removes edges to give a shorter walk.

At the end, we will have a walk from v_1 to v_2 with no repeated edges and no repeated vertices.

This is exactly the definition of a linear subgraph from v_1 to v_2.

Let v_1 be the vertex of G with the largest degree.

By Pigeonhole principle $d(v_1) \geq 6$.

looking at the remaining 9 vertices

$$\sum_{v \in V(G) \setminus \{v_1\}} d(v) = 56 - d(v_1) \geq 47$$

because $d(v_1) \leq 9$.

Therefore by pigeonhole principle on the remaining 9 vertices, there must be one with degree at least 6 $d(v_2) \geq 6$.

Therefore $d(v_1) + d(v_2) \geq 12$.

Among the remaining 8 vertices, there are ~~edges~~ at least 12 edges connecting to v_1 ~~and~~ or v_2

At most one connects v_1 to v_2.

~~[illegible]~~

Let $\{v_{i_1}, \dots, v_{i_k}\}$ be the vertices connected to v_1 that are not v_2

and let $\{v_{j_1}, \dots, v_{j_\ell}\}$ be the vertices connected to v_2 that are not v_1

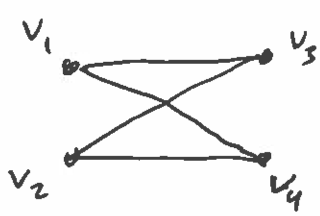
Then $|\{v_{i_1}, \dots, v_{i_k}\} \cap \{v_{j_1}, \dots, v_{j_\ell}\}| = k + \ell - \underbrace{|\{v_{i_1}, \dots, v_{i_k}\} \cup \{v_{j_1}, \dots, v_{j_\ell}\}|}_{\text{must be at most 8 vertices since cannot include } v_1 + v_2}$

$\geq 10 - 8$

≥ 2

So we can find 2 vertices $v_3, v_4 \in \{v_{i_1}, \dots, v_{i_k}\} \cap \{v_{j_1}, \dots, v_{j_\ell}\}$

so that v_3 and v_4 both connect to both v_1 and v_2 by edges in G:



Therefore there is a cycle of length 4 in G given by

$v_1 - v_3 - v_2 - v_4$

Extra Space:

Problem 8: Sam is scheduling a family vacation. The trip could last 5, 6 or 7 days. Based on the other things the family has scheduled during the summer, the trip must lie within the dates July 1 through July 22 (including July 1 and 22). How many different possibilities are there for the length and dates of the trip?

If the trip lasts 5 days it could start any ~~[illegible]~~ day between July 1 and July 18 $\rightsquigarrow$ 18 possibilities

If the trip lasts 6 days it could start any day between July 1 and July 17 $\rightsquigarrow$ 17 possibilities

If the trip lasts 7 days it could start any day between July 1 and July 16. $\longrightarrow$ 16 possibilities

Each trip is different so we add all these possibilities together to get

$18+17+16$ possibilities for the length + dates of the trip.

Problem 9: Suppose G is a connected planar graph, and there is a planar polygonal embedding such that every region has at least 5 edges in its sides. If G has 15 edges. What is the smallest number for the number of vertices of G? Give an example of a planar graph with this number of vertices and edges with the criterion that every region has at least 5 edges in its sides.

If G has a planar polygonal embedding and every region has at least 5 sides then

$$2\#E(G) \underset{\uparrow}{=} \sum_i \#\text{sides}(R_i) \geq 5\#R$$

R_i region

each ~~side~~ edge has 2 sides

So $5\#R \leq 2(15) \Rightarrow \#R \leq 6$

By Euler's formula ~~...~~ $\#V - \#E + \#R = 2$

So $\#V = \#E - \#R + 2 = 17 - \#R \geq 11$

Therefore the smallest number of vertices G can have is 11.

A graph with 11 vertices with a polygonal embedding so that each region has at least 5 sides is: