Practice Final Exam 2

Math 145, Spring 2019

Name: Solutions

Student ID: ____________________

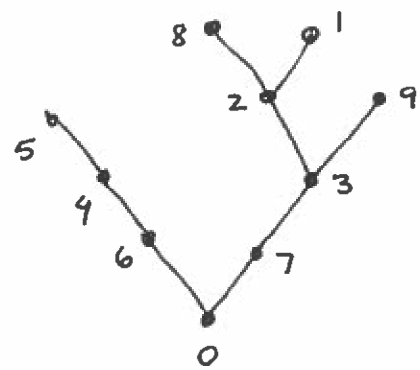
Every solution must contain an explanation written in words supporting your numerical solution to receive credit.

You do not need to simplify numerical expressions for your final answers (e.g. you can write $2^8 \cdot 4!$ instead of multiplying out to 6144.)

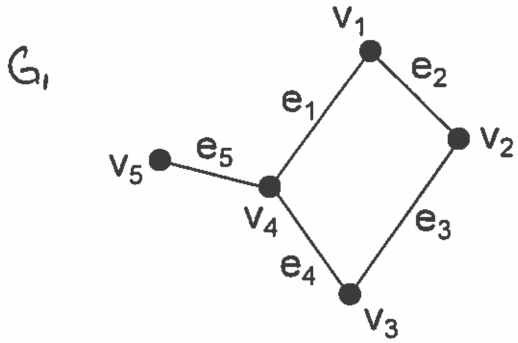
If you need extra space for your solutions, there is an extra page at the back of the exam. If you need extra space for any problem, write CONTINUED IN EXTRA SPACE on the page where the problem is given to you. In the extra space write the problem number that you are solving in that space.

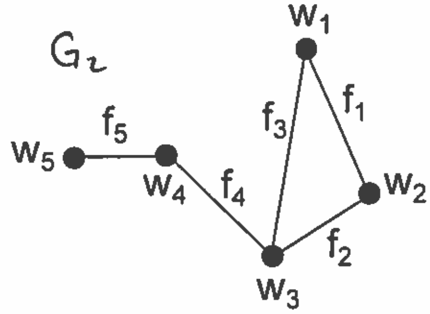
Problem 1: Construct the labeled tree that corresponds to the Prüfer code 24602337.

1	5	4	6	8	2	9	3	7
2	4	6	0	2	3	3	7	0



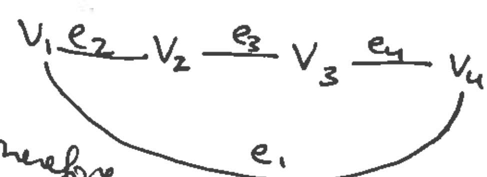
Problem 2: For the following pair of graphs, either prove they are isomorphic or prove they are not isomorphic.



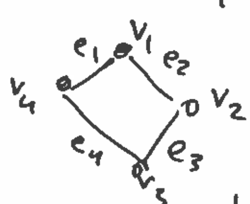


They are not isomorphic because the length of the only cycle in G_1 is 4 and ~~le~~ the length of the only cycle in G_2 is 3

The only cycle in G_1 is

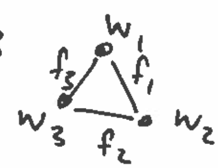


because v_5 has degree 1 & therefore cannot be in a cycle so any cycle must be a subgraph of



which is a cycle of length 4 and contains no closed ~~path~~ cycle of length 3, because any 3 edges form a linear graph which is not closed cycle.

G_2 has a cycle of length 3



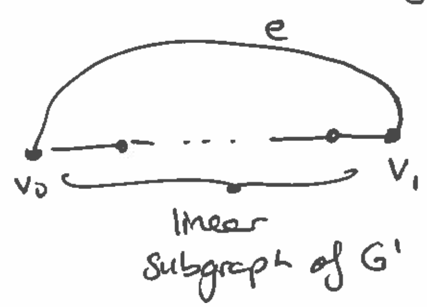
An isomorphism would take the length 3 cycle of G_2 to a length 3 cycle of G_1 but this does not exist.

Problem 3: Suppose G is a graph with no cycles. Let e be an edge in G. Let G' be the graph obtained from G by deleting the edge e. Prove that G' is not connected.

Let v_0 and v_1 be the vertices at the endpoints of e. We will show there is no linear subgraph of G' connecting v_0 to v_1.

For contradiction, Suppose there were a linear subgraph of G' connecting v_0 to v_1. Then this linear subgraph does not include the edge e since e is not in G'.

Then there is a subgraph of G obtained by adding e to the linear subgraph connecting v_0 to v_1 and this subgraph is a cycle in G contradicting the hypothesis.



Problem 4: A dance teacher is planning a dance recital for her 18 students. There will be 5 performances. Each student must dance in exactly one performance. Each performance must have at least one dancer. How many ways are there for the teacher to split up the students into the 5 different performances?

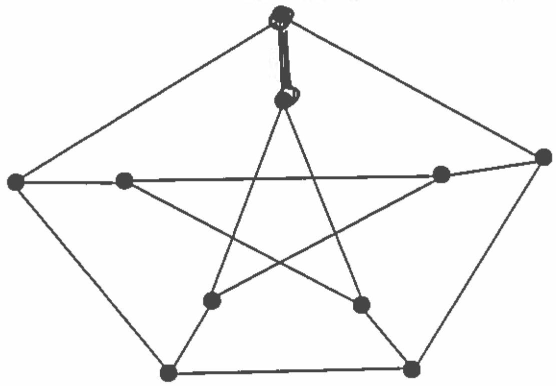
If we do not distinguish the students from each other then we need to divide up 18 students into 5 dances and it only matters how many students are in each dance. Then we can line up the dancers and choose where we split the dances:

There are 17 choices for the split points.

We need 4 split points to get 5 dances.

Therefore there are $\binom{17}{4}$ ways to split up the students into dances.

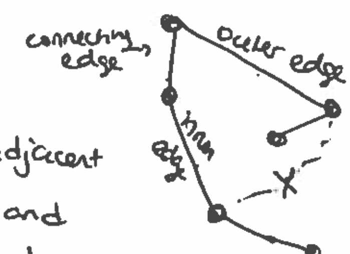
Problem 5: Prove that the following graph is not planar.



$\#V(G) = 10$

$\#E(G) = 15$

First we see that the shortest cycle in this graph is length 5. This is because the cycle around the outer 5 vertices is length 5, the cycle formed by the star of the inner 5 vertices is length 5, and if a cycle includes an edge connecting one of the outer vertices to an inner vertex then it must contain one outer edge and one inner edge adjacent to the connecting edge, and the resulting vertices at the endpoints are never connected by an edge so the cycle must have length greater than 4.



Suppose there were a planar embedding. Then every region must have at least 5 sides because the sides of a region form a cycle.

$$2\#E(G) = \sum_{R_i \text{ region}} \#\text{sides of } R_i \geq 5\#R \implies \#R \leq \tfrac{2}{5}\#E = 6$$

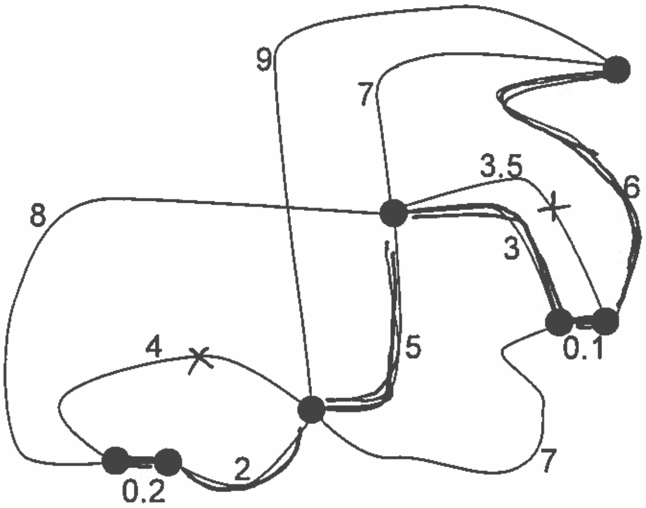
Euler's formula $\implies \#V(G) - \#E(G) + \#R(G) = 2$

$\implies 10 - 15 + \#R(G) = 2$

$\implies \#R(G) = 7$

but 7 is not less than or equal to 6 so we get a contradiction

Problem 6: Find the minimal cost spanning tree for the following weighted graph. Draw the spanning tree and determine its total cost.



Total cost:

$0.1+0.2+2+3+5+6$

$= 16.3$

Include 0.1

no cycle

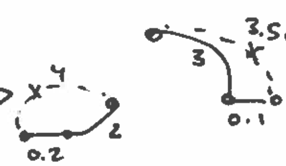
Include 0.2

no cycle

Include 2

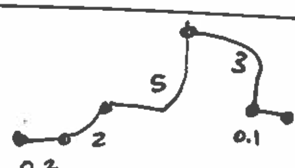
Include 3

Do not include 4 b/c creates a cycle

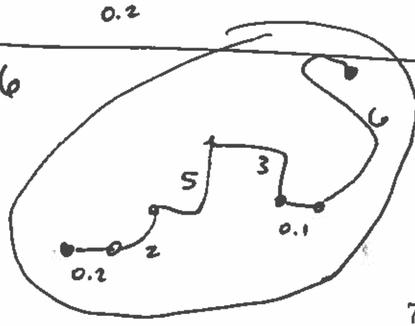


Do not include 3.5 b/c creates a cycle

Include 5



Include 6



Now all vertices are connected so any other edge will create a cycle.

Problem 7: How many subsets of $\{1,2,3,4,5,6,7,8,9,10\}$ contain *at least two* elements which are multiples of 3?

The multiples of 3 are 3, 6, 9

$A = \{\text{subsets containing } 3\}$

$B = \{\text{subsets containing } 6\}$

$C = \{\text{subsets containing } 9\}$

We want to count the number of elements in

$$(A \cap B) \cup (A \cap C) \cup (B \cap C)$$

$$|(A\cap B) \cup (A \cap C) \cup (B \cap C)| = |A\cap B| + |A \cap C| + |B \cap C| - 2\,|A \cap B \cap C|$$

because a subset gets counted once if it contains exactly 2 multiples of 3 in one of the counts $|A\cap B|$, $|A\cap C|$ or $|B \cap C|$ and no times in $|A\cap B\cap C|$

and it gets counted once in $|A\cap B|$, once in $|A\cap C|$, once in $|B\cap C|$ and once in $|A\cap B \cap C|$ if it contains all 3 multiples of 3

So in total it is counted $1+1+1-2\cdot 1 = 1$ time.

$|A \cap B| = 2^8$ because we decide yes/no whether to include 1, 2, 4, 5, 7, 8, 9, 10 and we always say yes to include 3 and 6

Similarly $|A\cap C| = 2^8$, $|B \cap C| = 2^8$

and $|A\cap B\cap C| = 2^7$ (choose yes/no to include 1, 2, 4, 5, 7, 8, 10)

Therefore there are

$$\boxed{2^8 + 2^8 + 2^8 - 2\cdot 2^7} = 2^8+2^8+2^8-2^8 = 2\cdot 2^8 = 2^9$$

subsets containing at least 2 multiples of 3.

Problem 8: Give the proof for the statement (directly from the definitions) that if T is a tree (a connected graph with no cycles) then T has a leaf (a vertex of degree 1).
with at least 2 vertices

Let v_0 be any vertex in T. (If we had a rooted tree, v_0 could be the root)

Take a walk starting at v_0 never repeating an edge, continuing until it is not possible to continue the walk.

First observe that the walk will never repeat a vertex because if we repeated a vertex, the walk would contain a cycle but T is a tree.

If the walk cannot continue, this means that we are at a vertex where all the edges adjacent to that vertex have already been used in the walk.

Since we never repeat a vertex, this means that after traveling along one edge ending at this vertex, we have used up all the edges adjacent to the vertex so the vertex has degree 1.

Problem 9: A simple graph G is called *regular* if every vertex has the same degree. Suppose G is a connected simple graph with 22 edges which is regular. How many vertices can G have?

Suppose G is regular and the degree of every vertex is d.

The degree - edge formula gives us

$$2\#E(G) = \sum_{v \in V(G)} d(v)$$

$\#E(G) = 22$ and $d(v) = d$ for every v

both integers so must be factors of 44

$\Rightarrow 44 = 2 \cdot 22 = \#V(G) \cdot d$

Possible $\#V(G)$:

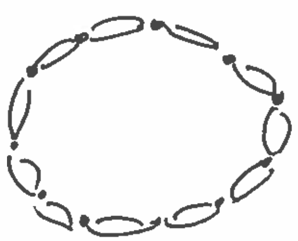
1 ← one vertex with 22 self edges

2 ← 2 vertices with 22 edges connecting

4 ← 11 edges

11

22 ← a cycle with 22 vertices + edges



44 ←

22 pairs

(Giving an example of a regular graph for each possible value of $V(G)$)