# Practice Midterm 1

Math 145, Spring 2019

**Name:** Solutions

**Every solution must contain an explanation written in words supporting your numerical solution to receive credit.**

**If you need extra space for your solutions, there is an extra page at the back of the exam. If you need extra space for any problem, write CONTINUED IN EXTRA SPACE on the page where the problem is given to you. In the extra space write the problem number that you are solving in that space.**

**Problem 1:** An hair dresser is planning which days she will cut her clients' hair. She needs to choose 20 out of the 30 days in April to work cutting her clients' hair. She does not want to work more than 1 of the 4 Sundays in April. How many ways are there for her to schedule her working days with clients, with the constraint that she will not work more than one Sunday?

There are 30 days in April and 4 Sundays.

The possible schedulles split into two cases:

Case 1:

The schedule includes exactly one of the Sundays.

Then there are 4 choices for the 1 Sunday

and $26 = 30-4$ choices for the other 19 working days.

Therefore there are a total of

$\binom{4}{1} \cdot \binom{26}{19}$ possible schedules in case 1.

Case 2: The schedule includes no Sundays.

Then there are $26 = 30-4$ possible choices for the 20 working days

So there are $\binom{26}{20}$ possible schedules in case 2.

There are no schedules in both case 1 and case 2.

In total there are $\binom{4}{1}\binom{26}{19} + \binom{26}{20} - 0 = \boxed{4\left(\frac{26!}{7!\,19!}\right) + \frac{26!}{6!\,20!}}$

possible ways for her to plan her work schedule.

**Problem 2:** How many numbers from 1 and 100 are a multiple of at least one of the numbers 3, 5, 7?

Let $A = \{\text{multiples of } 3 \text{ between } 1 + 100\} = \{3\cdot 1, 3\cdot 2, \ldots, \overset{99}{\overset{\shortparallel}{3\cdot 33}}\}$

$B = \{\text{multiples of } 5 \text{ between } 1 + 100\} = \{5\cdot 1, 5\cdot 2, \ldots, \overset{100}{\overset{\shortparallel}{5\cdot 20}}\}$

$C = \{\text{multiples of } 7 \text{ between } 1 + 100\} = \{7\cdot 1, 7\cdot 2, \ldots, \overset{98}{\overset{\shortparallel}{7\cdot 14}}\}$

$7\cdot 14 = 7\cdot 7\cdot 2 = 49\cdot 2 = 98$

Then $|A| = 33$

$|B| = 20$ as indicated by calculations

$|C| = 14$

$A\cap B = \{\text{multiples of } 3 \text{ \underline{and} } 5 \text{ btwn } 1 + 100\} = \{\text{multiples of } 15 \text{ btwn } 1+100\}$

$= \{15\cdot 1, 15\cdot 2, \ldots, \underset{90}{\underset{\shortparallel}{15\cdot 6}}\}$

$A\cap C = \{\text{multiples of } 3 \text{ \underline{and} } 7 \text{ btwn } 1+100\} = \{\text{multiples of } 21 \text{ btwn } 1+100\}$

$= \{21\cdot 1, 21\cdot 2, \ldots, \underset{84}{\underset{\shortparallel}{21\cdot 4}}\}$

$B\cap C = \{\text{multiples of } 5 \text{ \underline{and} } 7 \text{ btwn } 1+100\} = \{\text{multiples of } 35 \text{ btwn } 1+100\}$

$= \{35\cdot 1, 35\cdot 2\}$

So $|A\cap B| = 6$

$|A\cap C| = 4$

$|B\cap C| = 2$

Finally, $A\cap B\cap C = \{\text{multiples of } 3 \text{ \underline{and} } 5 \text{ \underline{and} } 7 \text{ btwn } 1+100\} = \{\text{multiples of } 105 \text{ btwn } 1+100\} = \emptyset$

By the inclusion exclusion principle

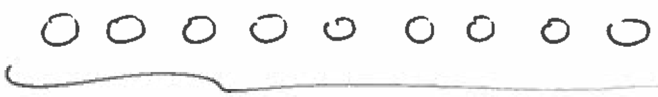
$|A\cup B\cup C| = |A| + |B| + |C| - |A\cap B| - |A\cap C| - |B\cap C| + |A\cap B\cap C|$

$= 33 + 20 + 14 - 6 - 4 - 2 + 0$

$= \boxed{55}$

and $A\cup B\cup C$ is the collection of numbers btwn 1 + 100 which are multiples of at least one of the numbers 3, 5, 7

**Problem 3:** A new cookie company is giving out 400 cookies for free to people passing by. Each person can take as many cookies as they want (possibly no cookies). At the end of the day, if there are any cookies left, the baker eats the rest. If 100 people pass by during the day, how many different ways are there for the cookies to be distributed?



400 cookies distributed to 100 + 1 people

customers ↗ ↖ baker

~~Idea~~ If we order the people by the time they arrive, we just need to determine where the split points are between the 400 cookies.

Since some of the people may not want any cookies, some of the split points may occur in the same position.

There are 401 positions for the split points (including the positions before the first cookie & after the last cookie)

Let $s_1, s_2, \ldots, s_{100}$ be the 100 split points where $s_i$ divides the cookies for the $i^{th}$ person from the cookies for the $(i+1)^{th}$ person.

Let $t_1, t_2, \ldots, t_{100}$ be given by the formula

$$(*)\begin{cases} t_1 = s_1 \\ t_2 = s_2 + 1 \\ \vdots \\ t_i = s_i + (i-1) \end{cases}$$

Then $1 \le s_1 \le s_2 \le \ldots \le s_{100} \le 401$

$$\Updownarrow$$

$1 \le t_1 < t_2 < \ldots < t_{100} \le 500$

The number of possible choices of $\{(s_1, s_2, \ldots, s_{100}) \mid 1 \le s_1 \le s_2 \le \ldots \le s_{100} \le 401\}$ is the same as the number of $\{(t_1, t_2, \ldots, t_{100}) \mid 1 \le t_1 < t_2 < \ldots < t_{100} \le 500\}$ because they are in 1-1 correspondence by the bijection $(*)$. The number of $t_1, \ldots, t_{100}$ is $\binom{500}{100}$ because there is no repetition. So the number of cookie distributions is $\binom{500}{100}$.

**Problem 4:** There are $n$ students in class. We know that $k$ are in their $1^{st}$ year at UC Davis and $m$ are in their $2^{nd}$ year at UC Davis. Show that both sides of the following equation give ways of counting the number of ways that the students can be split up into $1^{st}$ year students, $2^{nd}$ year students, and students who have been at UC Davis for at least 3 years.

$$\underbrace{\binom{n}{k}\binom{n-k}{m}}_{LHS} = \underbrace{\binom{n}{n-k-m}\binom{k+m}{m}}_{RHS}$$

LHS: $\binom{n}{k}$ is the number of ways of choosing which k students of all n students are $1^{st}$ years

If we take out those $1^{st}$ years there are n-k students left and so the number of ways to chose m of them as $2^{nd}$ years is $\binom{n-k}{m}$

Therefore there are $\binom{n}{k}\cdot\binom{n-k}{m}$ total ways of distributing which students are the k $1^{st}$ years and which are the m $2^{nd}$ years.
(The rest are $3^{rd}$ year or above)

RHS: If there are k $1^{st}$ years and m $2^{nd}$ years there are n-k-m students in $3^{rd}$ year or above.

There are $\binom{n}{n-k-m}$ ways of choosing which n-k-m students of the n total students are in $3^{rd}$ year or above.

Once we remove the n-k-m students in $3^{rd}$ year or above, there are k+m students left. m of them are $2^{nd}$ years so there are $\binom{k+m}{m}$ ways of choosing which are the $2^{nd}$ years

& Once we know who is in $3^{rd}$ year or above & who is in $2^{nd}$ year, the remaining k must be in their $1^{st}$ year.

Therefore there are $\binom{n}{n-k-m}\cdot\binom{k+m}{m}$ ways of determining who is in $1^{st}$, $2^{nd}$, and $3^{rd}$+ years.

**Problem 5:** Suppose you have a ladder with $n$ rungs. At every step you can go up 1 rung at a time, 2 rungs at a time, or 3 rungs at a time. Define the correct number of initial conditions and a recurrence formula to determine the numbers $L_n$ that count the number of different ways to climb the ladder.

$L_1$ :  one rung : only one way to go up : take 1 1rung step

$\Rightarrow \underline{L_1 = 1}$

$L_2$ :  2 rungs : 2 ways to go up :
- take 2 1rung steps
- take 1 2rung step

$\Rightarrow \underline{L_2 = 2}$

$L_3$ :  3 rungs : 4 ways to go up
- take 3 1rung steps
- take 1 2rung step then 1 1rung step
- take 1 1rung step then 1 2rung step
- take 1 3rung step

$\underline{L_3 = 4}$

$L_n$ : <u>Case 1:</u> First step is a 1rung step: then there are n-1 rungs left so the number of ways to go up the rest is $L_{n-1}$

<u>Case 2:</u> First step is a 2 rung step: then there are n-2 rungs left so the number of ways to go up the rest is $L_{n-2}$

<u>Case 3:</u> First step is a 3 rung step : then there are n-3 rungs left so the number of ways to go up the rest is $L_{n-3}$

$\Rightarrow L_n = L_{n-1} + L_{n-2} + L_{n-3}$ ← recursive formula

$L_1 = 1, \quad L_2 = 2, \quad L_3 = 4$ ← initial conditions