Practice Midterm 2

Math 145, Spring 2019

Name: Solutions

Every solution must contain an explanation written in words supporting your numerical solution to receive credit.

If you need extra space for your solutions, there is an extra page at the back of the exam. If you need extra space for any problem, write CONTINUED IN EXTRA SPACE on the page where the problem is given to you. In the extra space write the problem number that you are solving in that space.

Problem 1: How many positive integers are there with *exactly* four digits such that all of the digits are different?

__ __ __ __

If a positive integer has exactly 4 digits, the first digit is one of the numbers $\{1, 2, 3, \ldots, 9\}$ and the second, third, & fourth one one of the numbers $\{0, 1, 2, \ldots, 9\}$

If all the digits are different, we have a dependent ordered problem, but the options 0 is not allowed for the first digit. To deal with this, we split into two cases:

Case 1: 0 is not one of the digits

Then this is a normal dependent ordered problen so there are $\underline{9 \cdot 8 \cdot 7 \cdot 6}$ possibilities.

Case 2: 0 is one of the last 3 digits. This splits into 3 similar subcases:

Case 2A: 0 is the 2^{nd} digit __ $\underline{0}$ __ __

Then we have 9 choices for digit 1, 8 for digit 3, and 7 for digit 4

So there are $\underline{9 \cdot 8 \cdot 7}$ possibilities.

Case 2B: 0 is the 3^{rd} digit.

Similarly there are $9 \cdot 8 \cdot 7$ possibilities for the 1^{st}, 2^{nd} & 4^{th} digits.

Case 2C: 0 is the 4^{th} digit

Similarly there are $9 \cdot 8 \cdot 7$ possibilities for the 1^{st}, 2^{nd}, & 3^{rd} digits.

In total there are: $9 \cdot 8 \cdot 7 \cdot 6 + 3 \cdot 9 \cdot 8 \cdot 7$ different positive integers with exactly 4 distinct digits.

$(3024) + (1512) = \overset{2}{4}536$

Problem 2: How many subsets of $S = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ are there which do not contain 2 or do not contain 5?

Let $A = \{$subsets that do not contain 2$\}$

Let $B = \{$subsets that do not contain 5$\}$

We are trying to count the number of subsets in $A \cup B$
because these are the subsets that do not contain 2 or subsets do not contain 5.

$|A| = 2^8$ because for each element 1, 3, 4, 5, 6, 7, 8, 9 we choose yes or no is the element in the subset (for 2 we must choose no).

Similarly

$|B| = 2^8$ because for each element 1, 2, 3, 4, 6, 7, 8, 9 we choose yes or no is the element in the subset (for 5 we must choose no).

$A \cap B = \{$subsets that do not contain 2 and do not contain 5$\}$

$|A \cap B| = 2^7$ because for each element 1, 3, 4, 6, 7, 8, 9 we choose yes or no is the element in the subset (for 2 and 5 we must choose no)

Therefore $\boxed{|A \cup B| = 2^8 + 2^8 - 2^7}$ by the inclusion exclusion principle

$\uparrow$
# of subsets of S that do not contain 2 or do not contain 5.

Problem 3: At a grand opening of a bicycle store, 100 people put their names in a lottery for free bicycle prizes. The store will give out 8 helmets, 5 lights, and 3 locks. How many different outcomes are there for winners of the different bike prizes?

In total $8+5+3=16$ people will receive prizes out of the 100 people.

The number of ways to choose the people who win prizes is $\binom{100}{16} = \frac{100!}{(100-16)!\,16!} = \frac{100!}{84!\,16!}$

Of the 16 prize winners, 8 get the same helmets, 5 get the same lights, and 3 get the same locks.

The number of ways to rearrange the helmet, light, and lock prizes among the 16 prize winners is then

$$\frac{16!}{8!\,5!\,3!}$$

$16!$ ← # reorderings of 16 things

$8!\,5!\,3!$ ← # of reorderings that do not change the outcome because they switch helmets with helmets lights with lights and locks with locks

So in total there are

$$\frac{100!}{84!\,16!} \cdot \frac{16!}{8!\,5!\,3!}$$

different ~~cea~~ outcomes.

Problem 4: Show that both sides of the following equation count the number of binary strings of 0's and 1's of length n which have at most k 1's where $0 \le k \le n-1$:

$$\underbrace{\sum_{j=0}^{k} \binom{n}{j}}_{LHS} = \underbrace{\sum_{j=0}^{k} \binom{n-1-j}{k-j} 2^j}_{RHS}$$

LHS: The number of binary strings of length n with exactly j 1's is determined by where we choose to put the j 1's out of the n possible choices. Therefore there are $\binom{n}{j}$ binary strings with exactly j 1's so there are

$$\sum_{j=0}^{k} \binom{n}{j} \text{ binary strings with } 0 \le j \le k \text{ 1's.}$$

RHS: If there are at most $k \le n-1$ 1's then there are at least $n-k$ 0's, and $n-k \ge 1$. Let's consider where the $n-k^{th}$ 0 ends up (counting the 0's in the string from left to right). If the $n-k^{th}$ 0 ends up in the $(n-j)^{th}$ position then there are $n-k-1$ 0's to the left of the $n-j^{th}$ position and any number of 0's to the right of the $n-j^{th}$ position:

$$\underbrace{\underset{1}{\_}\ \underset{2}{\_}\ \_\ \cdots\ \underset{n-j-1}{\_}}_{\text{exactly } n-k-1 \text{ 0's}}\ \underset{n-j}{0}\ \underbrace{\underset{n-j+1}{\_}\ \cdots\ \underset{n}{\_}}_{\text{anything}}$$

There are $\binom{n-j-1}{n-k-1}$ ways of choosing where the 0's in the first $n-j-1$ spots go and 2^j ways of choosing the string in the last j spots.

$0 \le j \le k$ because $n-k \le n-j \le n$ (the $n-j^{th}$ position has the $n-k^{th}$ 0)

So all possibilities are enumerated by $\sum_{j=0}^{k} \binom{n-j-1}{n-k-1} 2^j$

Problem 5: Let F_n denote the n^{th} Fibonacci number defined by the initial values $F_1 = F_2 = 1$ and the recurrence relation $F_n = F_{n-1} + F_{n-2}$.

Prove that for all $n \geq 1$,

$$-F_1 + F_2 - F_3 + \cdots - F_{2n-1} + F_{2n} = F_{2n-1} - 1$$

Proof by induction:

Base case: $n=1$ Want to show: $-F_1 + F_2 = F_1 - 1$

$F_1 = 1, F_2 = 1 \Rightarrow -F_1 + F_2 = -1 + 1 = 0 = 1 - 1 = F_1 - 1$. ✓

Inductive hypothesis: $-F_1 + F_2 - \ldots - F_{2(n-1)-1} + F_{2(n-1)} = F_{2(n-1)-1} - 1$

$\Rightarrow -F_1 + F_2 \ldots - F_{2n-3} + F_{2n-2} = F_{2n-3} - 1$

$\Rightarrow -F_1 + F_2 - \ldots - F_{2n-3} + F_{2n-2} - F_{2n-1} + F_{2n} = (F_{2n-3} - 1) - F_{2n-1} + F_{2n}$

By the recursive formula: $F_{2n} = F_{2n-1} + F_{2n-2}$

and $F_{2n-1} = F_{2n-2} + F_{2n-3}$

$\Rightarrow (F_{2n-3} - 1) - F_{2n-1} + F_{2n} = F_{2n-3} - (F_{2n-2} + F_{2n-3}) + (F_{2n-1} + F_{2n-2}) - 1$

$= \cancel{F_{2n-3}} - \cancel{F_{2n-3}} \cancel{- F_{2n-2} + F_{2n-2}} + F_{2n-1} - 1 = F_{2n-1} - 1$

$\Rightarrow -F_1 + F_2 - \ldots - F_{2n-1} + F_{2n} = F_{2n-1} - 1$ as claimed

$\square$