Homework 1

Math 145, Spring 2019

Every solution must contain an explanation written in words supporting your numerical solution to receive credit.

1 Counting

1. In a game, you roll one 6 sided dice, one 8 sided dice, and one 20 sided dice. How many different possible outcomes are there for the resulting roll? (We consider each dice differently so if the six sided dice rolls a 1, the eight sided dice rolls a 2, and the 20 sided dice rolls a 3, this is considered a different outcome from the situation where the six sided dice rolls a 2, the eight sided dice rolls a 1, and the 20 sided dice rolls a 3.) Explain how you got your answer in words.

2. Ten people arrive early to buy tickets for a concert. They form an ordered line. How many ways are there for the 10 different people to stand in the line? Explain how you got your answer in words.

3. Your friend has 8 shiny new quarters which are indistinguishable from each other. She flips all 8 of them and leaves them face up on the table in front of you. You cannot tell the difference between the different quarters or the order in which they were flipped. How many possible different resulting states can there be of the 8 quarters on the table? Explain how you got your answer in words.

4. Alice and Bob are team captains for a soccer match. There are 12 other kids who want to play. Alice and Bob take turns choosing the remaining six kids for their team in order based on the position that will be filled. (For example, we consider the resulting states different if Alice chooses Sam as her first pick versus choosing Sam as her third pick.) How many different ways are there for all of the positions on the two teams to be filled? Explain how you got your answer in words.

5. Caroline and Doug are team captains for a dodgeball match. There are 10 other kids who want to play. Caroline and Doug take turns choosing kids for their team so at the end they each choose five kids for their team. The order in which kids were chosen does not affect the game. The only thing that matters is which team they are on. How many possible ways are there to split up the 10 kids into Caroline and Doug's teams? Explain how you got your answer in words.

6. A tennis tournament has 12 participating teams. Each match faces two teams against each other. If team A faces team B in a match, we consider this the same pairing as if team B faces team A. If team A face team B on the first court, we consider this the same pairing as if team A faces team B on the fourth court. How many ways are there to pair up the 12 teams into 6 pairings? Explain how you got your answer in words.

2 Sets

7. True or false?

   (a) x is an element of the set $\{\{x, y\}, \{z\}, 4\}$.

   (b) $\{\{z\}, 4\}$ is a subset of the set $\{\{x, y\}, \{z\}, 4\}$.

   (c) The integers are a subset of the real numbers.

8. Prove that if A, B, and C are sets then $(A \cup B) \cup C = A \cup (B \cup C)$.

9. Prove that if A, B, and C are sets then $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

10. There are seven gymnasts at a competition: Amanda, Dorothy, Gabby, Jacquelyn, Kelly, Madison, and Simone. One is chosen for the gold medal, one for silver, and one for bronze. What are the number of possibilities for the winning gold, silver, and bronze medalists? Explain how this problem can be understood as a question of the number of *ordered* k-element subsets of a set S with n elements. What is the set S? What are k and n?

11. Let $S = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$. How many (unordered) subsets does S have where the size of the subset (number of elements of the subset) is any *positive even number*? Explain how you got your answer in words.

3 Induction

12. Prove by induction that the sum of the first n squares $(1 + 4 + 9 + \cdots + n^2)$ is $n(n + 1)(2n + 1)/6$.

13. Prove by induction that

$$2^0 + 2^1 + \cdots + 2^{n-1} = 2^n - 1.$$

14. Let $a_0 = 1$ and let $a_{m+1} = 2a_m + 1$ for all positive integers $m \geq 0$. Find an explicit formula for a_m (in terms of m only) and prove your formula is correct.

15. Suppose you have a square piece of paper, and after 1 minute you cut it into four squares. After the second minute you cut one of those squares into four squares. Every minute you cut one square of paper into four squares. Prove that after n minutes, the number of squares you have is $3n + 1$.