Homework 2

Math 145, Spring 2019

Every solution must contain an explanation written in words supporting your numerical solution to receive credit.

1 Inclusion-Exclusion

1. A travel survey asks participants which cities they have visited amongst New York, Chicago, and San Francisco. Using the spreadsheet where the data is collected, they determine the following numbers

NY	Chi	SF	NY $\cap$ Chi	NY $\cap$ SF	Chi $\cap$ SF	NY $\cap$ Chi $\cap$ SF
300	200	250	120	150	100	60

   (a) How many people were surveyed who have gone to at least one of the three cities?

   (b) How many people were surveyed who have gone to at least *two* of the three cities?

2. Consider the set $S = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$. How many subsets contain at least one of the elements 3 or 7?

3. How many positive integers less than 999 are NOT a multiple of 5 and NOT a multiple of 7? Explain how you got your answer with words.

4. An elementary school has three sports teams. Suppose that in one class it is true that for any two students in the class, there is at least one team such that those two students are both members of that team. Prove that there is a team that contains at least 2/3 of the students in that class.

5. How many positive integers are there in the range from 1 to $1,000,000$ that are either perfect squares or perfect cubes or both? Explain how you got your answer with words.

6. A survey of the students in Math 145 found that 15 of the students are currently taking a physics class, 25 are currently taking a computer science class, and 50 are taking another math class. There is only one student who is taking all three: physics, computer science, and another math class, and there are 7 students who are not taking any of the three. There are 70 total students in the class.

   (a) If P is the set of students taking a physics class, C is the set taking a computer science class, and M is the set taking another math class, what is the value of $|P \cap C| + |P \cap M| + |C \cap M|$?

   (b) How many students are taking *at least* two of the three types of classes surveyed? Explain how you got your answer with words.

   (c) How many students are taking *exactly* two of the three types of classes surveyed? Explain how you got your answer with words.

2 Pigeonhole Principle

7. Suppose there are 10 lines in the plane and no two of them are parallel. Suppose that at every point in the plane there are at most 3 lines passing through that point. Show that if we fix one of the 10 lines L_0, then there are at least 5 points on L_0 where other lines intersect it.

8. Suppose you have a subset X of the natural numbers $\mathbb{N}$ with $n+1$ elements ($X \subset \mathbb{N}$, $|X| = n+1$).

   (a) Show that there exist two elements $x, y \in X$ such that $x - y \geq n$.

   (b) Show that there exist two elements $x, y \in X$ such that $x - y$ is *divisible* by n.

9. Suppose there is a square dartboard that is 8 inches by 8 inches. If you throw 65 darts at the board (and hit the board every time), prove that at least 2 of them will be less than 1.5 inches apart. (You may use the fact that the longest distance between two points in a 1×1 square is $\sqrt{2} \approx 1.41$).