Homework 3

Math 145, Spring 2019

Every solution must contain an explanation written in words supporting your numerical solution to receive credit.

1 Binomial Theorem and binomial coefficients (3.1)

1. Suppose we have n people in a school. The student representative council is a group of k of these people. One of the k members of the student representative council is the president. The rest of the members have no named position, they are just members.

   (a) Prove that the number of ways to choose the student representative council and its president is $k\binom{n}{k}$.

   (b) Use this idea to prove the following identity:

   $$\sum_{k=1}^{n} k\binom{n}{k} = n2^{n-1}.$$

2. Suppose we have n people in a school. The student representative council is a group of k of these people. Of the k people on the student representative council, a small collection of s of them are on the executive council.

   (a) How many ways are there to choose the student representative council and the executive council (note that every member of the executive council must be a member of the student representative council)? Express this problem in terms of sets

   (b) Use this counting problem to give a combinatorial proof of the following identity. For all positive integers n, k, s with $s \leq k \leq n$

   $$\binom{n}{k}\binom{k}{s} = \binom{n}{s}\binom{n-s}{k-s}$$

3. Prove the identity

   $$3^n = 2^n\binom{n}{0} + 2^{n-1}\binom{n}{1} + 2^{n-2}\binom{n}{2} + \cdots + 2^1\binom{n}{n-1} + \binom{n}{n}.$$

4. Prove the identity

$$\binom{n}{0}^2 + \binom{n}{1}^2 + \binom{n}{2}^2 + \cdots + \binom{n}{n-1}^2 + \binom{n}{n}^2 = \binom{2n}{n}.$$

5. Prove the identity

$$\binom{n}{0}\binom{m}{k} + \binom{n}{1}\binom{m}{k-1} + \cdots + \binom{n}{k-1}\binom{m}{1} + \binom{n}{k}\binom{m}{0} = \binom{n+m}{k}$$

2 Permutations, multisets (3.2-3.4)

6. How many different words can you invent by rearranging the letters in MATHEMATICS ? (They do not need to be real words, but they must use the same letters repeated the same number of times.)

7. A waiter takes the order of a big table of 10 people at a diner for breakfast. 4 of them order fried eggs with bacon, 2 order pancakes, 2 order identical omelettes, 1 orders waffles, and 1 orders french toast. The waiter puts in the order but completely forgets who ordered what dish. How many ways are there for the plates of food to be distributed to the 10 people? What are the chances that the waiter delivers the right dish to every person?

8. In a lottery, 6 balls are drawn randomly from a set of 49 balls numbered $1, 2, \ldots, 49$. You choose 6 numbers on your ticket, and you win if you have the same 6 numbers as the numbers of the balls that are drawn. It does not matter if your numbers are in the same order as the order the balls are drawn (order does NOT matter). We consider two different lottery rules in the following two parts.

   (a) How many different possibilities are there for the lottery drawing if the balls are *not replaced* after they are drawn (so all 6 numbers must be different)?

   (b) How many different possibilities are there for the lottery drawing if the balls *are replaced* after they are drawn (so you can repeat numbers)?

9. A bag contains blue balls and red balls.

   (a) If the bag has exactly 50 blue balls and exactly 50 red balls for a total of 100 balls, and you draw 10 balls randomly from the bag (without replacing), what is the probability that you will draw exactly 5 red balls and exactly 5 blue balls?

   (b) If the bag has exactly $2n$ red balls and $2n$ blue balls for a total of $4n$ balls, and you draw $2n$ balls randomly, what is the probability that you drew exactly n red balls and n blue balls? (Find a formula in terms of n.)