- Defn - open cover, finite subcover
- Examples + non examples $\mathbb{R}$, $[0,1]$, $(0,1)$
- Compact subspace
- Continuous image of cpt is cpt

Lecture 21

Compact Spaces

On a topological space (X,τ) the topology τ tells us the open sets on X.

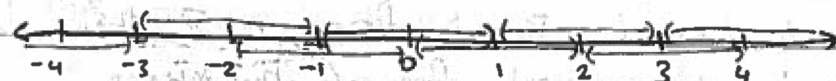
Defn: An open cover of (X,τ) is a collection of open sets $\{U_\alpha\}_{\alpha\in I}$ ~~[illegible]~~ such that

$$\bigcup_{\alpha\in I} U_\alpha = X.$$

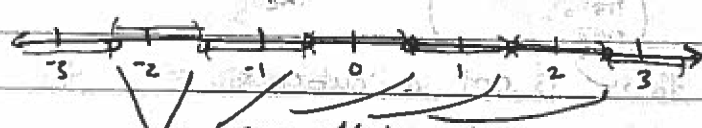
Examples: $(X,\tau) = (\mathbb{R}, \tau_{Euc})$

$\{B_1(n)\}_{n\in\mathbb{Z}}$ is an open cover of $\mathbb{R}$ because every real number $x\in\mathbb{R}$ is distance <1 from $\lfloor x\rfloor \in \mathbb{Z}$

balls of radius 1 centred at integers ($B_1(n) = (n-1, n+1)$)



but $\{B_{1/2}(n)\}_{n\in\mathbb{Z}}$ is not an open cover



misses all the 1/2 integers

$\{B_1(x)\}_{x\in\mathbb{R}}$ is an open cover

$\{B_\varepsilon(x)\}_{x\in\mathbb{R},\ \varepsilon>0}$ is an open cover

$\{\mathbb{R}\}$ is an open cover

$\{(-\infty,1), (0,\infty)\}$ is an open cover

A subcover is a subset of the collection of open sets such that this subcollection is still an open cover.

e.g. $\{B_1(n)\}_{n\in\mathbb{Z}}$ is a subcover of $\{B_1(x)\}_{x\in\mathbb{R}}$

$\star$ $(\mathbb{R}, \tau_{Zar})$ compact?

Defn: A topological space (X, τ) is compact if every open cover of (X, τ) has a finite subcover.

This quantifier makes it very hard to prove something is compact, but easy to show something is not compact.

Thm: $(\mathbb{R}, \tau_{st})$ is not compact.

Pf: $\{B_1(n)\}_{n\in\mathbb{Z}}$ is an open cover of $\mathbb{R}$

If $\mathbb{R}$ were compact there would be a finite subcover $\{B_1(n_1), \dots, B_1(n_k)\}$ such that

$$B_1(n_1) \cup \dots \cup B_1(n_k) = \mathbb{R}$$

but if $N = \max\{n_1, \dots, n_k\}$ then

$N+1 \in \mathbb{R}$ but $N+1 \notin B_1(n_1) \cup \dots \cup B_1(n_k)$

so this cannot be a cover. ∎

In fact $\{B_1(n)\}$ has no proper subcover:

If we remove $B_1(n_0)$ for any $n_0 \in \mathbb{Z}$ then n_0 has distance ≥ 1 from all other integers

so $n_0 \notin \left(\bigcup_{\substack{n \neq n_0 \\ n\in\mathbb{Z}}} B_1(n) \right)$

so this is not a subcover.

Note: Some open covers of $\mathbb{R}$ do have finite subcovers:

eg. $\{(-\infty, 1), (0, \infty)\}$ is finite itself so it is itself a finite subcover.

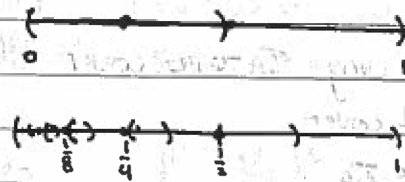
$\{(-\infty,1), (0,\infty)\} \cup \{B_1(n)\}_{n\in\mathbb{Z}}$ is an open cover with a finite subcover $\{(-\infty,1), (0, \infty)\}$

Every space X has a finite cover $\{X\}$

Compact means every open cover has a finite subcover. Not compact means we can find at least one open cover with no finite subcover.

Thm: The open interval $(0,1)$ is not compact.

Pf: Consider the open cover



$$\{(\tfrac{1}{2},1)\} \cup \{B_{\frac{1}{2^{n+1}}}(\tfrac{1}{2^n})\}_{n=1}^{\infty}$$

$\boxed{A}$ This is an open cover because for each $x \in (0,1)$:

- If $x > \frac{1}{2}$ $\quad x \in (\frac{1}{2},1)$
- If $0 < x < \frac{1}{2}$ then $\exists N$ s.t. $\frac{1}{2^{N+1}} < x < \frac{1}{2^N}$

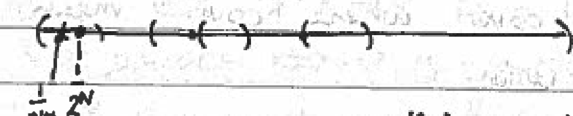
  and then $x \in B_{\frac{1}{2^{N+1}}}(\frac{1}{2^N})$

$\boxed{B}$ This has no finite subcover because:

if we take a finite subcollection

$(\frac{1}{2},1),\ B_{\frac{1}{2^{n_1+1}}}(\frac{1}{2^{n_1}}), \ldots, B_{\frac{1}{2^{n_k+1}}}(\frac{1}{2^{n_k}})$

let $N = \max\{n_1, \ldots, n_k\}$ then $\frac{1}{2^{N+1}} \in (0,1)$ and

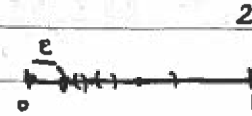


$\frac{1}{2^{N+1}} \notin (\frac{1}{2},1) \cup B_{\frac{1}{2^{n_1+1}}}(\frac{1}{2^{n_1}}) \cup \ldots \cup B_{\frac{1}{2^{n_k+1}}}(\frac{1}{2^{n_k}})$

so this cannot cover $(0,1)$.

Again in fact we cannot remove anything from the cover because $\frac{1}{2^k} \in B_{\frac{1}{2^{k+1}}}(\frac{1}{2^k})$ but not in any other $B_{\frac{1}{2^{n+1}}}(\frac{1}{2^n})$ $n \neq k$. ☐

We will show $[0,1]$ is compact



but proving this is fairly difficult.

When is a subspace compact?

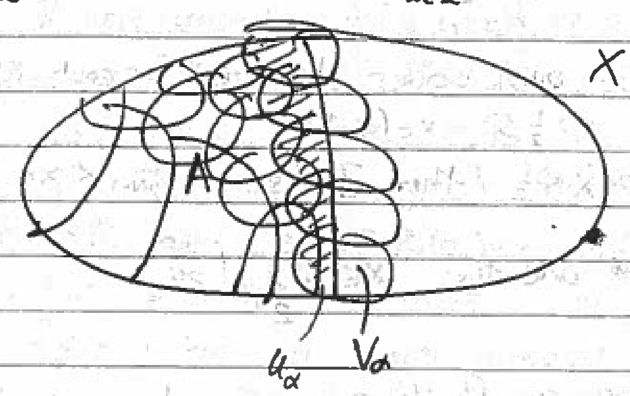
~~Recall~~ (X, τ_X) topological space

$A \subset X$ τ_A subspace topology.

A is compact iff for every τ_A-open cover $\{U_\alpha\}_{\alpha \in I}$ there is a finite subcover.

For each α, $U_\alpha = V_\alpha \cap A$ for $V_\alpha \in \tau_A$

$\bigcup_{\alpha \in I} U_\alpha = A$ so $\bigcup_{\alpha \in I} V_\alpha \supseteq A$



$\exists$ Finite subcover of $\{U_\alpha\}$ $\Leftrightarrow$ $U_{\alpha_1} \cup \cdots \cup U_{\alpha_N} = A$

$\Leftrightarrow$ $V_{\alpha_1} \cup \ldots \cup V_{\alpha_N} \supseteq A$

So $A \subset X$ is compact iff every τ_X-open cover of A has a finite subcover where "cover" means A is contained in the union.