Extra Practice for Midterm 2

Math 147, Fall 2018

Name:

Problem 1: Let $(\mathbb{R}, \tau_{Euc})$ be the real line with the Euclidean topology.

(a) Give an example of a subspace A of $\mathbb{R}$ where A has the subspace topology τ_A, and a subset $U \subset A$ such that the interior of U as a subset of A is the *same* as the interior of U as a subset of $\mathbb{R}$.

$A = (0, 2)$, $U = (0, 1)$. U is open in $\mathbb{R}$ and $U = A \cap U$ is open in A so its interior in $\mathbb{R}$ is itself and its interior in A is also itself.

(b) Give an example of a subspace B of $\mathbb{R}$ where A has the subspace topology τ_B, and a subset $V \subset B$ such that the interior of B as a subset of B is *different* than the interior of V as a subset of $\mathbb{R}$.

$B = [0, 2]$, $U = (1, 2]$. The open interval $(1, 3)$ is open in $\mathbb{R}$ and $U = (1, 3) \cap B$ so U is open in B. Therefore the interior of U as a subset of B is U itself. However, the interior of U in $\mathbb{R}$ is $(1, 2)$ because 2 is not an interior point of U in the topology on $\mathbb{R}$ since any open subset of $\mathbb{R}$ containing 2 contains $(2 - \varepsilon, 2 + \varepsilon)$ for some $\varepsilon > 0$ and so it contains $2 + \varepsilon/2$ which is not contained in U. $(1, 2)$ is open in $\mathbb{R}$ so each point in $(1, 2)$ is in the interior of U as a subset of $\mathbb{R}$.

Problem 2: Let (X, τ_X) and (Y, τ_Y) be regular spaces. Prove that the product $(X \times Y, \tau_X \times \tau_Y)$ is regular. [Hint: use the equivalent definition of regularity, that a space is regular if and only if for every point x and open set U containing x, there is an open subset V such that $x \in V$ and $\overline{V} \subset U$.]

Let $(x, y) \in X \times Y$ be any point and U be an open subset of $X \times Y$ such that $(x, y) \in U$. Since a basis for the product topology is given by subsets of the form $U_1 \times U_2$ where $U_1 \subset X$ is open in τ_X and $U_2 \subset Y$ is open in τ_Y, there exists such a $U_1 \times U_2$ such that $(x, y) \in U_1 \times U_2 \subset U$. Then since X is regular and $x \in U_1$, there exists an open subset $V_1 \subset X$ such that $x \in V_1$ and $\overline{V_1} \subset U_1$. Similarly, since Y is regular and $y \in U_2$, there exists an open subset $V_2 \subset Y$ such that $y \in V_2$ and $\overline{V_2} \subset U_2$. Therefore $V_1 \times V_2$ is an open subset of $X \times Y$ such that $(x, y) \in V_1 \times V_2$ and $\overline{V_1} \times \overline{V_2} \subset U_1 \times U_2 \subset U$. Since $\overline{V_1 \times V_2} = \overline{V_1} \times \overline{V_2}$, $V_1 \times V_2$ provides the necessary open subset to verify the criterion to ensure that $X \times Y$ is regular.

To prove that $\overline{V_1 \times V_2} = \overline{V_1} \times \overline{V_2}$, let (a, b) be a limit point of $V_1 \times V_2$. Then for any open subset $W_1 \subset X$ such that $a \in W_1$ and any open subset $W_2 \subset Y$ such that $b \in W_2$, $W_1 \times W_2$ is an open subset of $X \times Y$ containing (a, b) so $W_1 \times W_2 \cap V_1 \times V_2 \neq \emptyset$. Therefore there exists a point $(c, d) \in W_1 \times W_2 \cap V_1 \times V_2$ so $W_1 \cap V_1 \neq \emptyset$ since $c \in W_1 \times V_1$ and $W_2 \cap V_2 \neq \emptyset$ because $d \in W_2 \cap V_2$. Therefore a is a limit point of V_1 and b is a limit point of V_2 so $\overline{V_1 \times V_2} \subset \overline{V_1} \times \overline{V_2}$.

For the other direction suppose a is a limit point of V_1 and b is a limit point of V_2, we will try to show (a, b) is a limit point of $V_1 \times V_2$. Let $U \subset X \times Y$ be an open subset containing (a, b). Then by the definition of the basis for the product topology there exists $U_1 \subset X$ and $U_2 \subset Y$ open in their respective topologies such that $(a, b) \in U_1 \times U_2 \subset U$. Then $a \in U_1$ so $U_1 \cap V_1 \neq \emptyset$ and $b \in U_2$ so $U_2 \cap V_2 \neq \emptyset$. Therefore $(U_1 \times U_2) \cap (V_1 \times V_2) \neq \emptyset$ so (a, b) is a limit point of $V_1 \times V_2$, so $\overline{V_1} \times \overline{V_2} \subset \overline{V_1 \times V_2}$.

Problem 3: Let X be the set of natural numbers $1, 2, 3, \cdots$. We define a topology τ' on X as follows. $U \subset X$ is open in τ' if and only if either one or both of the following two conditions hold

1. $X \setminus U$ is finite
2. $1 \in X \setminus U$

Prove that X with the topology τ' is *not connected*.

Let $A = X \setminus \{2\}$. Then A is open because $X \setminus A$ is finite (only the number 2). $X \setminus A = \{2\}$ is open because $1 \in X \setminus (X \setminus A)$. Therefore A is also closed. $A \neq \emptyset$ and $A \neq X$ so X is not connected.

Problem 4: Let $X = \mathbb{R}^2$ with the Euclidean topology. Define an equivalence relation $\sim$ on X by $(x_1, y_1) \sim (x_2, y_2)$ if and only if $x_1 + 2y_1^2 = x_2 + 2y_2^2$. Let $Y = \mathbb{R}$ with the Euclidean topology. Construct a map $f : X/_\sim \to Y$ and show that f is *well-defined, continuous and has an inverse.* You do NOT need to prove that f^{-1} is continuous (f^{-1} probably will be continuous, you just do not need to prove it).

Let $f([(x, y)]) = x + 2y^2$.

f **is well defined:** If $[(x_1, y_1)] = [(x_2, y_2)]$ then $x_1 + 2y_1^2 = x_2 + 2y_2^2$ so $f([(x_1, y_1)]) = f([(x_2, y_2)])$.

f **is continuous:** Let $p : X \to X/_\sim$ be the quotient map. Then $f \circ p(x, y) = x + 2y^2$ is a polynomial map so it is continuous. Therefore $f : X/_\sim \to Y$ is continuous.

f **has an inverse:** Let $f^{-1} : Y \to X/_\sim$ be defined by $f^{-1}(s) = [(s, 0)]$.

$$f(f^{-1}(s)) = f([(s, 0)]) = s$$

$$f^{-1}(f([(x, y)])) = f^{-1}(x + 2y^2) = [(x + 2y^2, 0)]$$

and $[(x, y)] = [(x + 2y^2, 0)]$ because $x + 2y^2 = x + 2y^2 + 0^2$.

Therefore f^{-1} is an inverse function for f.