Extra Practice for Midterm 1

Math 147, Fall 2018

Problem 1: Suppose $f : X \to Y$ is a function between sets X and Y. Suppose $\mathcal{B}_Y$ is a basis for a topology on Y.

(a) Show that if we define the collection

$$\mathcal{B}_X = \{f^{-1}(B) \mid B \in \mathcal{B}_Y\}$$

then $\mathcal{B}_X$ is a basis on X.

The first axiom for a basis: Let $x \in X$. Then $f(x) \in Y$. Since $\mathcal{B}_Y$ is a basis, there exists $B \in \mathcal{B}_Y$ such that $f(x) \in B$. Therefore $x \in f^{-1}(B) \in \mathcal{B}_X$.

The second axiom for a basis: Let $x \in f^{-1}(B_1) \cap f^{-1}(B_2)$ for $B_1, B_2 \in \mathcal{B}_Y$. Then $f(x) \in B_1 \cap B_2$. Since $B_1, B_2 \in \mathcal{B}_Y$, which is a basis, there exists $B_3 \in \mathcal{B}_Y$ such that $f(x) \in B_3 \subset B_1 \cap B_2$. Then $x \in f^{-1}(B_3) \subset f^{-1}(B_1) \cap f^{-1}(B_2)$ so we prove the second basis axiom for $\mathcal{B}_X$.

(b) If we give X the topology τ_X generated by the basis $\mathcal{B}_X$ and Y the topology generated by the basis $\mathcal{B}_Y$, show that $f : (X, \tau_X) \to (Y, \tau_Y)$ is a continuous function.

It suffices to check that the preimage of each basis element of $\mathcal{B}_Y$ is open in the topology τ_X. If $B \in \mathcal{B}_Y$ then $f^{-1}(B) \in \mathcal{B}_X$. Basis elements are always open in the topology generated by the basis because if $u \in f^{-1}(B)$ then $u \in f^{-1}(B) \subseteq f^{-1}(B)$. Therefore f is continuous.

Problem 2: Let d_X be a metric on a set X and d_Y be a metric on a set Y.

(a) Show that the function $d : (X \times Y) \times (X \times Y) \to \mathbb{R}$ defined by

$$d((x_1, y_1), (x_2, y_2)) = \max\{d_X(x_1, x_2), d_Y(y_1, y_2)\}$$

is a metric on $X \times Y$.

(1) Positive definite: Since $d_X(x_1, x_2) \geq 0$ and $d_Y(y_1, y_2) \geq 0$, $\max\{d_X(x_1, x_2), d_Y(y_1, y_2)\} \geq 0$. If $\max\{d_X(x_1, x_2), d_Y(y_1, y_2)\} = 0$ then $d_X(x_1, x_2) = 0$ and $d_Y(y_1, y_2) = 0$, therefore by positive definiteness of d_X and d_Y, $x_1 = x_2$ and $y_1 = y_2$ therefore $(x_1, y_1) = (x_2, y_2)$.

(2) Symmetric: Since d_X and d_Y are symmetric $d((x_1, y_1), (x_2, y_2)) = \max\{d_X(x_1, x_2), d_Y(y_1, y_2)\} = \max\{d_X(x_2, x_1), d_Y(y_2, y_1)\} = d((x_2, y_2), (x_1, y_1))$.

(3) Triangle inequality: Consider three points (x_1, y_1), (x_2, y_2) and (x_3, y_3). Now

$$d((x_1, y_1), (x_3, y_3)) = \max\{d_X(x_1, x_3), d_Y(y_1, y_3)\}$$

By the triangle inequality for d_X, we know that

$$\begin{aligned} d_X(x_1, x_3) &\leq d_X(x_1, x_2) + d_X(x_2, x_3) \\ &\leq \max\{d_X(x_1, x_2), d_Y(y_1, y_2)\} + \max\{d_X(x_2, x_3) + d_Y(y_2, y_3)\} \\ &= d((x_1, y_1), (x_2, y_2)) + d((x_2, y_2), (x_3, y_3)) \end{aligned}$$

and by the triangle inequality for d_Y we know that

$$\begin{aligned} d_Y(y_1, y_3) &\leq d_Y(y_1, y_2) + d_Y(y_2, y_3) \\ &\leq \max\{d_X(x_1, x_2), d_Y(y_1, y_2)\} + \max\{d_X(x_2, x_3) + d_Y(y_2, y_3)\} \\ &= d((x_1, y_1), (x_2, y_2)) + d((x_2, y_2), (x_3, y_3)) \end{aligned}$$

Since $d((x_1, y_1), (x_3, y_3)) = d_X(x_1, x_3)$ or $d_Y(y_1, y_3)$ and both are less than or equal to $d((x_1, y_1), (x_2, y_2)) + d((x_2, y_2), (x_3, y_3))$, the triangle inequality holds.

(b) Show that in this metric $B_\varepsilon((x, y)) = B_\varepsilon(x) \times B_\varepsilon(y)$.

Suppose $(a, b) \in B_\varepsilon((x, y))$ then $d((a, b), (x, y)) < \varepsilon$ so $\max\{d_X(a, x), d_Y(b, y)\} < \varepsilon$. Therefore $d_X(a, x) < \varepsilon$ and $d_Y(b, y) < \varepsilon$. Therefore $a \in B_\varepsilon(x)$ and $b \in B_\varepsilon(y)$ so $(a, b) \in B_\varepsilon(x) \times B_\varepsilon(y)$. We conclude that $B_\varepsilon((x, y)) \subseteq B_\varepsilon(x) \times B_\varepsilon(y)$.

For the other direction, suppose $(a, b) \in B_\varepsilon(x) \times B_\varepsilon(y)$. Then $d_X(a, x) < \varepsilon$ and $d_Y(b, y) < \varepsilon$. Therefore $\max\{d_X(a, x), d_Y(b, y)\} < \varepsilon$ so $d((a, b), (x, y)) < \varepsilon$ so $(a, b) \in B_\varepsilon((x, y))$. Therefore $B_\varepsilon(x) \times B_\varepsilon(y) \subseteq B_\varepsilon((x, y))$. Since we have both sets are subsets of each other they must be equal.

Problem 3: Let $f : (\mathbb{R}, \tau_1) \to (\mathbb{R}, \tau_2)$ be the function $f(x) = 2x$.

(a) If τ_1 is the discrete topology and τ_2 is the Euclidean topology, determine whether or not f is continuous and prove it.

f is continuous. For the proof, let $U \in \tau_2$ be an open subset. Then $f^{-1}(U) \subset \mathbb{R}$ is a subset of $\mathbb{R}$ so it is open in the discrete topology. Therefore the preimage of any open set is open so f is continuous.

(b) If τ_1 is the Euclidean topology and τ_2 is the discrete topology, determine whether or not f is continuous and prove it.

f is not continuous. To prove this, consider the set $\{1\} \subset \mathbb{R}$. This is open in the discrete topology because it is a subset. $f^{-1}(\{1\}) = \{\frac{1}{2}\}$. Since τ_1 is the Euclidean topology, $\{\frac{1}{2}\}$ is not an open subset because if $\varepsilon > 0$, $B_\varepsilon(\frac{1}{2})$ is not contained in $\{\frac{1}{2}\}$ because the ball contains other points such as $\frac{1}{2} + \frac{\varepsilon}{2}$. Therefore f is not continuous.

(c) If τ_1 is the trivial topology defined by $\tau_1 = \{\emptyset, \mathbb{R}\}$ and τ_2 is the Euclidean topology, determine whether or not f is continuous and prove it.

f is not continuous. To prove this, consider the open interval $(0, 1)$ in the Euclidean topology. Then $f^{-1}((0, 1)) = (0, \frac{1}{2})$. Since $(0, \frac{1}{2}) \neq \emptyset$ and $(0, \frac{1}{2}) \neq \mathbb{R}$, $(0, \frac{1}{2}) \notin \tau_2$. Therefore f is not continuous.

Problem 4: Let $A \subset X$ be a subset of a topological space. Show that $Bdry(A) = \emptyset$ if and only if A is both open and closed in X.

See solutions to Homework 4.