Practice Midterm 2

Math 147, Fall 2018

Name:

Problem 1: Let (X, τ_X) be a topological space. Let Y be a subspace of X with the subspace topology τ_Y. Let $A \subseteq Y$. Let $\overline{A}^X$ denote the closure of A in (X, τ_X) and let $\overline{A}^Y$ denote the closure of A in (Y, τ_Y).

(a) Prove that $\overline{A}^Y \subset \overline{A}^X$.

Suppose $p \in \overline{A}^Y$. Then either $p \in A$ in which case $p \in \overline{A}^X$ by definition of the closure, or p is a limit point of A in the subspace topology τ_Y on Y. Therefore for every open set $V \in \tau_Y$ such that $p \in V$, $V \cap A \neq \emptyset$. We will show that p is a limit point of A in the topology τ_X on X. Let $U \in \tau_X$ be an open subset of X such that $p \in U$. Then $V = Y \cap U$ is an open subset of Y: $V \in \tau_Y$. Therefore $V \cap A \neq \emptyset$. Since $V \subseteq U$, $U \cap A \neq \emptyset$. Therefore p is a limit point of A in τ_X so $p \in \overline{A}^X$.

(b) Give an example where $\overline{A}^Y \neq \overline{A}^X$.

Consider $X = \mathbb{R}$ where τ_X is the Euclidean topology and $Y = (0, 2)$. Let $A = (1, 2)$. Then $\overline{A}^Y = [1, 2)$ and $\overline{A}^X = [1, 2]$ because 2 is a limit point of A in $\mathbb{R}$ but $2 \notin Y$ so it cannot be a limit point of A in Y with the subspace topology.

Problem 2: Let $X = \mathbb{R} \setminus \{0\} = (-\infty, 0) \cup (0, \infty)$ with the Euclidean topology τ. Define an equivalence relation $\sim$ on X by $x_1 \sim x_2$ if and only if $x_1 = cx_2$ where $c > 0$. Let Y be the set with two elements $Y = \{+, -\}$. Let τ_Y be the discrete topology on Y. Construct a map $f : X/_\sim \to Y$ and show that f is *well-defined, continuous and has an inverse.* You do NOT need to prove that f^{-1} is continuous (f^{-1} probably will be continuous, you just do not need to prove it).

Define a map $f : X/_\sim \to Y$ by $f(x) = +$ if $x > 0$ and $f(x) = -$ if $x < 0$.

f **is well defined:** If $x_1 = cx_2$ for $c > 0$ then $x_1 > 0$ if and only if $x_2 > 0$ (and $x_1 < 0$ if and only if $x_2 < 0$) since multiplication by a positive number does not change the sign of a number. Therefore if $[x_1] = [x_2]$ and $x_1 > 0$, $f([x_1]) = + = f([x_2])$, and if $[x_1] = [x_2]$ and $x_1 < 0$ then $f([x_1]) = - = f([x_2])$.

f **is continuous:** $Y = \{+, -\}$ with the discrete topology, so its open subsets are $\emptyset$, $\{+\}$, $\{-\}$, and $\{-, +\}$. From the fact that $f : X/_\sim \to Y$ is a function, we know that $f^{-1}(\emptyset) = \emptyset$ and $f^{-1}(Y) = X/_\sim$ so these preimages are open. Therefore, to show that f is continuous, we just need to check that $f^{-1}(\{+\})$ and $f^{-1}(\{-\})$ are open sets in $X/_\sim$.

By definition of f, $f^{-1}(\{+\}) = \{[x] \mid x > 0\}$. By definition of the quotient topology $\{[x] \mid x > 0\}$ is an open subset of $X/_\sim$ if and only if $p^{-1}(\{[x] \mid x > 0\})$ is open in X. $p^{-1}(\{[x] \mid x > 0\}) = (0, \infty)$ which is open in $\mathbb{R}$ with the Euclidean topology so it is open in $\mathbb{R} \setminus \{0\}$.

Similarly, $f^{-1}(\{-\}) = \{[x] \mid x < 0\}$ and $p^{-1}(\{[x] \mid x < 0\}) = (-\infty, 0)$. Since $(-\infty, 0)$ is open in $\mathbb{R} \setminus \{0\}$, $f^{-1}(\{-\})$ is open in $X/_\sim$ so f is continuous.

f **has an inverse:** Define $f^{-1} : Y \to \mathbb{R} \setminus \{0\}$ by setting $f^{-1}(+) = [1]$ and $f^{-1}(-) = [-1]$.

Then $f(f^{-1}(+)) = f([1]) = +$ and $f(f^{-1}(-)) = f([-1]) = -$.

If $x > 0$, $f^{-1}(f([x])) = f^{-1}(+) = [1]$ and $[1] = [x]$ because $x = x \cdot 1$ so letting $c = x$, using the fact that $c = x > 0$, we have $x \sim 1$.

If $x < 0$, $f^{-1}(f([x])) = f^{-1}(-) = [-1]$ and $[-1] = [x]$ because $x = (-x) \cdot (-1)$ so letting $c = -x$ and using the fact that $c = -x > 0$, we have $x \sim -1$.

Therefore f and f^{-1} are inverse functions.

Problem 3: If X is a metric space, and τ is the topology induced by the metric (with basis the open balls of positive radius), show that (X, τ) is Hausdorff.

Let $x_1, x_2 \in X$ where $x_1 \neq x_2$. Then $d(x_1, x_2) > 0$. Let $\varepsilon = d(x_1, x_2)$. Let $U_1 = B_{\varepsilon/4}(x_1)$ and $U_2 = B_{\varepsilon/4}(x_2)$. Then $x_1 \in U_1$, $x_2 \in U_2$, U_1 and U_2 are open in the topology induced by the metric since they are basis elements. We will check that $U_1 \cap U_2 = \emptyset$. This is because if $y \in U_1 \cap U_2$ then $d(x_1, y) < \varepsilon/4$ and $d(x_2, y) < \varepsilon/4$, therefore by the triangle inequality:

$$\varepsilon = d(x_1, x_2) \leq d(x_1, y) + d(y, x_2) < \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \frac{\varepsilon}{2}$$

which would imply that $\varepsilon < \varepsilon/2$ which is a contradiction so $U_1 \cap U_2 = \emptyset$ so (X, τ) is Hausdorff.

Problem 4: Let A and B be subsets of a topological space (X, τ) such that (A, τ_A) and (B, τ_B) are connected with the subspace topology. If $A \cap B \neq \emptyset$ show that $A \cup B$ with the subspace topology is connected.

Suppose for contradiction that $A \cup B$ is not connected. Then there exists a subset $C \subset A \cup B$ such that C is open and closed in $A \cup B$ with the subspace topology, and $C \neq \emptyset$, $C \neq A \cup B$. Let $C_1 = A \cap C$ and $C_2 = B \cap C$. Then since a set is open in the subspace topology on A as a subset of X if and only if it is open in the subspace topology on A as a subset of $A \cup B$, C_1 is open in the subspace topology on A. Similarly C_2 is open in the subspace topology on B.

Since C is closed in $A \cup B$, $D = (A \cup B) \setminus C$ is open so $D_1 = A \cap D$ is open in A and $D_2 = B \cap D$ is open in B. $A \setminus C_1 = D_1$ because $x \in D_1$ if and only if $x \in A$ and $x \in (A \cup B) \setminus C$, which is true if and only if $x \in A$ and $x \notin C$ so $D_1 = A \setminus C_1$. Since D_1 is open, C_1 is closed. Similarly, since D_2 is open and $A \setminus C_2 = D_2$, C_2 is closed.

Therefore C_1 is open and closed in A so since A is connected, $C_1 = \emptyset$ or $C_1 = A$. Similarly since C_2 is open and closed in B and B is connected, $C_2 = \emptyset$ or $C_2 = B$.

If $C_1 = \emptyset$ then $C \subseteq B$ so $C = C_2$. Since $C \neq \emptyset$, $C = B$, but $A \cap B \neq \emptyset$ so $C \cap A \neq \emptyset$ which contradicts the assumption that $C_1 = \emptyset$.

Therefore $C_1 = A$. Since $A \cap B \neq \emptyset$, $C_2 \neq \emptyset$, therefore $C_2 = B$. Thus $C = C_1 \cup C_2 = A \cup B$ which contradicts the original assumption that $C \neq A \cup B$.

Therefore $A \cup B$ is connected.