MATH 147 FINAL REFERENCE SHEET

Here is a reference/reminder of some facts from set theory that you may need. You may use these facts freely on the exam.

(1) If $A \subseteq X$ then

$$X \setminus (X \setminus A) = A.$$

(2) If $U_\alpha \subset X$ for all $\alpha \in \mathcal{I}$, DeMorgan's rules say that

$$X \setminus \left(\bigcup_{\alpha \in \mathcal{I}} U_\alpha \right) = \bigcap_{\alpha \in \mathcal{I}} (X \setminus U_\alpha)$$

$$X \setminus \left(\bigcap_{\alpha \in \mathcal{I}} U_\alpha \right) = \bigcup_{\alpha \in \mathcal{I}} (X \setminus U_\alpha)$$

(3) If $f : X \to Y$ is a function

$$f^{-1}(Y) = X$$

and

$$f^{-1}(\emptyset) = \emptyset.$$

(4) If $f : X \to Y$ is a function and $V_\alpha \subset Y$ for all $\alpha \in \mathcal{I}$ then

$$f^{-1} \left(\bigcup_{\alpha \in \mathcal{I}} V_\alpha \right) = \bigcup_{\alpha \in \mathcal{I}} f^{-1}(V_\alpha)$$

and

$$f^{-1} \left(\bigcap_{\alpha \in \mathcal{I}} V_\alpha \right) = \bigcap_{\alpha \in \mathcal{I}} f^{-1}(V_\alpha)$$

Here are some theorems from the end of the course. You may use the statement of these theorems (as well as any other theorems we have proven in class) in any of the problems. If you are asked to prove one of these theorems, you should provide a proof. If you are using the theorem to prove another result, you do not need to give a proof of these theorems.

Theorem 1. *Let (X, τ) be a compact space and $C \subset X$ a closed subset. Then C is compact.*

Theorem 2. *Let (X, τ) be compact and $K \subset X$ a subset with infinitely many different points. Then K has at least one limit point $x \in X$ such that for every open set U such that $x \in U$, $U \cap K \setminus \{x\} \neq \emptyset$.*

Theorem 3. *Let (X, τ) be a Hausdorff space. If $F \subset X$ is a compact subset, then F is closed.*

Theorem 4. *Let (X, τ_X) be compact and (Y, τ_Y) be Hausdorff. If $f : (X, \tau_X) \to (Y, \tau_Y)$ is a continuous function which is bijective, then f^{-1} is continuous so f is a homeomorphism.*

Note: $f : X \to Y$ is bijective if and only if it has an inverse function $f^{-1} : Y \to X$ such that $f(f^{-1}(y)) = y$ for all $y \in Y$ and $f^{-1}(f(x)) = x$ for all $x \in X$. (This is a statement about sets, and says nothing about continuity.)

Theorem 5. *If (X, τ_X) and (Y, τ_Y) are compact spaces, and $\tau_{X\times Y}$ denotes the product topology, then $(X \times Y, \tau_{X\times Y})$ is compact.*

Theorem 6. *If $A \subset \mathbb{R}^n$ is a subset of $\mathbb{R}^n$, then A is compact if and only if A is closed and bounded.*

Theorem 7. *If X is Hausdorff, K is any subset, and x is a limit point of K such that for every open set U where $x \in U$, $U \cap K \setminus \{x\} = \emptyset$, then x is an accumulation point of K (every open set U such that $x \in U$ has the property that $U \cap K$ contains infinitely many points).*

Theorem 8. *If (X, d) is a metric space where* every *infinite subset has an accumulation point in X then X with the metric topology is compact.*

We proved this with two lemmas:

Lemma 9. *If (X, d) is a metric space such that every infinite subset has an accumulation point in X, then for any $\varepsilon > 0$, the open cover $\{B_\varepsilon(x)\}_{x\in X}$ has a finite subcover $B_\varepsilon(x_1) \cup \cdots B_\varepsilon(x_n) \supset X$.*

Lemma 10. *If (X, d) is a metric space such that every infinite subset has an accumulation point in X, then for each open cover $\{U_\alpha\}_{\alpha\in\mathcal{I}}$, of X, there exists $\varepsilon > 0$ such that for each $x \in X$, there exists $\alpha_0 \in \mathcal{I}$ such that $B_\varepsilon(x) \subset U_{\alpha_0}$.*