Compactness overview

① Open covers $\{U_\alpha\}_{\alpha\in I}$ ← a bunch of open sets s.t. $X \subseteq \bigcup_{\alpha\in I} U_\alpha$

Finite subcover: $\{U_{\alpha_1}, \dots, U_{\alpha_m}\} \subseteq \{U_\alpha\}_{\alpha\in I}$ ← subcollection of

finitely many open sets

still covers: $X \subseteq U_{\alpha_1} \cup \dots \cup U_{\alpha_m}$

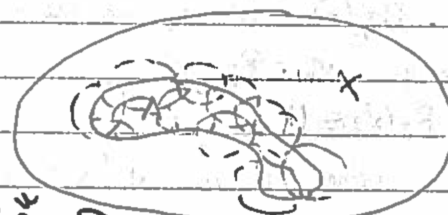
② Non compact things: $\mathbb{R}$, $(0,1)$

open cover: $\{(n-1, n+1)\}_{n\in\mathbb{Z}}$

Integer $n\in\mathbb{Z}$ is only in $(n-1,n+1)$

so if we remove any open set from the cover it will not be a cover anymore.

③ Compact subsets:



Open cover: $\{V_\alpha\}_{\alpha\in I}$ $V_\alpha \overset{\subseteq X}{} $ open in X

$A \subseteq \bigcup_{\alpha\in I} V_\alpha$

or $U_\alpha = A \cap V_\alpha$ $\{U_\alpha\}_{\alpha\in I}$ $U_\alpha \overset{\subseteq A}{}$ open in (subspace topology)

$A \underset{(=)}{\subseteq} \bigcup_{\alpha\in I} U_\alpha$

Try to come up with the proofs of these on your own (check notes or try book if you get stuck)

Thm 1: If X compact and A closed then A is compact

$A \subseteq X$, X compact = "A bounded"

Thm 2: If X ~~compact~~ Hausdorff, A compact $\Rightarrow A$ closed.

Need Hausdorff: $(\mathbb{R}, \tau_{Zariski})$ NOT Hausdorff

$(0,1)$ NOT closed in τ_{Zar}

but $((0,1), \tau_{sub})$ is still compact

by same proof that $\mathbb{R}, \tau_{Zar}$ is compact

④ Compactness in $\mathbb{R}^n$

$\mathbb{R}$: • $[a,b]$ is compact

• $A \subset \mathbb{R}$ is compact $\Leftrightarrow$ A is closed and bounded (i.e. $A \subseteq [a,b]$ for some $a,b \in \mathbb{R}$)

$\mathbb{R}^n$: • $[a_1,b_1] \times \dots \times [a_n,b_n]$ is compact

• $A \subset \mathbb{R}^n$ is compact $\Leftrightarrow$ A is closed and bounded (bounded = $A \subseteq [a_1,b_1] \times \dots \times [a_n,b_n]$ for some $a_1,b_1,\dots,a_n,b_n \in \mathbb{R}$)

⑤ Compactness and limit/accumulation points; metric spaces

Thm: If (X,τ) compact, every infinite subset $K \subset X$ has a limit point ($x \in X$ s.t. for every open $U \ni x$, $(U \cap K) \setminus \{x\} \neq \emptyset$)

Lemma: If X is Hausdorff, any limit point^x of a set K is an accumulation point

i.e. for every open $U \ni x$ not only is $(U \cap K) \setminus \{x\}$ non-empty, it is actually infinite.

Thm: ~~Every~~ A metric space (X,d) has the property that

• every infinite subset $K \subset X$ has an accumulation point

if and only if

• X is compact

⑥ Compactness + continuity:

Thm: $f: X \to Y$ continuous, $A \subseteq X$ compact $\Rightarrow f(A)$ compact

Thm: $f: X \to Y$ continuous, bijective, X compact

Open Sets

- Metric spaces : U open if $\forall x \in U \ \exists r > 0$ s.t. $B_r(x) \subseteq U$
- In terms of a topology: U open if $U \in \tau$
- In terms of a basis: U open iff $\forall x \in U \ \exists B \in \mathcal{B}$ s.t. $x \in B \subseteq U$

  U open iff $U = \bigcup_{\alpha \in I} B_\alpha$ where $B_\alpha \in \mathcal{B} \ \forall \alpha \in I$
- U open iff $U = \mathrm{Int}(U)$: $z \in \mathrm{Int}(U) \Leftrightarrow \exists$ open set V s.t. $z \in V \subseteq U$.
- Union of arbitrarily many open sets is open
- Finite intersection of open sets is open

Closed Sets

- C is closed in $(X, \tau) \Leftrightarrow X \setminus C$ is open in (X, τ)
- ~~Metric spaces:~~ C is closed $\Leftrightarrow$ C contains all its limit points
- C closed $\Leftrightarrow$ $C = \overline{C}$ ← closure
- $\overline{C}$ = intersection of all closed sets containing C

  $= C \cup \{\text{limit points of } C\}$
- Arbitrary intersection of closed sets is closed
- Finite union of closed sets is closed

Continuity $f : (X, \tau_X) \to (Y, \tau_Y)$

- For every open set $U \subset Y$, $f^{-1}(U)$ is open in X.
- Metric Spaces: For every point $x \in X$, and every $\varepsilon > 0$ there exists $\delta > 0$ such that $f(B_\delta(x)) \subseteq B_\varepsilon(f(x))$

  * Whenever $x_n \to x$, $f(x_n) \to f(x)$
- If $\mathcal{B}_Y$ is a basis for τ_Y and $f^{-1}(B)$ is open in X for every $B \in \mathcal{B}_Y$, then f is continuous.
- If $p : X \to X/\sim$ is a quotient map and $f : X/\sim \to Y$ is a map then f is continuous $\Leftrightarrow$ $f \circ p : X \to Y$ is continuous
- If $f : X \to Y$ is continuous and bijective and X is compact and Y is Hausdorff then f^{-1} is continuous.
- f continuous $\Leftrightarrow$ for every closed set $C \subset Y$, $f^{-1}(C) \subset X$ is closed.

Metric: $\cdot\ d(x,y) > 0$ if $x \neq y$ $\quad d(x,x) = 0$

$\cdot\ d(x,y) = d(y,x)$

$\cdot\ d(x,z) \leq d(x,y) + d(y,z)$

subspace topology: (X, τ_X) $A \subset X$ $\quad \tau_{sub}$ subspace topology on A

$U \in \tau_{sub} \Leftrightarrow U = V \cap A$ for some $V \in \tau_X$

- If $\mathcal{B}_X$ is a basis for τ_X

$\mathcal{B}_A = \{ B \cap A \mid B \in \mathcal{B}_X \}$ is a basis for τ_{sub}

- If (X, τ_X) is Hausdorff/regular/normal, (A, τ_{sub}) is Hausdorff/regular/normal

Product topology: $(X, \tau_X), (Y, \tau_Y) \rightarrow (X \times Y, \tau_{X \times Y})$

Basis for $\tau_{X \times Y}$: $\{ U \times V \mid U \in \tau_X, V \in \tau_Y \}$

Not every open set in $\tau_{X \times Y}$ has the form $U \times V$ but

every open set is a union of these $\bigcup_{\alpha} U_\alpha \times V_\alpha$

- If (X, τ_X) and (Y, τ_Y) are connected, $(X \times Y, \tau_{X \times Y})$ is connected
- If (X, τ_X) and (Y, τ_Y) are compact, $(X \times Y, \tau_{X \times Y})$ is compact.

Quotient topology: $\sim$ equivalence relation on X, $X/\sim$ is the set of equivalence classes. the quotient topology τ_{quot} on $X/\sim$ induced by the topology τ_X on X is defined by: let $p: X \to X/\sim$ be $p(x) = [x]$, $U \in \tau_{quot} \Leftrightarrow p^{-1}(U) \in \tau_X$.

Separation properties:

- Hausdorff: $\forall x, y \in X\ \exists\ U, V$ open s.t. $x \in U$, $y \in V$ and $U \cap V = \emptyset$.
- Regular: $\forall x \in X$, $C \subset X$ closed, $x \notin C$, $\exists\ U, V$ open s.t. $x \in U$, $C \subset V$, $U \cap V = \emptyset$.

  Equivalent to: $\forall x \in X$ and U open s.t. $x \in U$, $\exists\ V$ open s.t. $x \in V$, $\bar{V} \subset U$.
- Normal: $\forall$ closed sets $C_1, C_2 \subseteq X$ s.t. $C_1 \cap C_2 = \emptyset$ $\exists\ U, V$ open s.t.

  $C_1 \subset U$, $C_2 \subset V$ and $U \cap V = \emptyset$.

Connectedness

- (X,τ) connected $\Leftrightarrow$ the only subsets $A \subset X$ which are open and closed are $A=\emptyset$ and $A=X$.
- (X,τ) not connected $\Leftrightarrow$ $\exists$ closed sets $F, G \subset X$ s.t. $F \cup G = X$ and $F \cap G = \emptyset$.
- (X,τ) connected $\Leftrightarrow$ Every continuous map $f:(X,\tau)\to(\{0,1\}, \tau_{dis})$ is constant

  (So if there is a nonconstant continuous map, (X,τ) is not connected)
- If $f:(X,\tau_X)\to(Y,\tau_Y)$ is a homeomorphism then (X,τ_X) is connected $\Leftrightarrow$ (Y,τ_Y) is connected.
- If $f:(X,\tau_X)\to(Y,\tau_Y)$ is continuous and $A\subset X$ is a connected subset then $f(A)$ is a connected subset of Y.
- In $(\mathbb{R},\tau_{Euc})$ $A \subset (\mathbb{R}, \tau_{Euclidean})$ is connected $\Leftrightarrow$ A is an interval meaning if $a,b\in A$ then for any $a<c<b$, $c\in A$.
- Locally connected: every point x has an open set U s.t. $x \in U$, U connected.
- Path connected: For every $x,y\in X$ there exists a continuous $\gamma:[0,1]\to X$ such that $\gamma(0)=x$, $\gamma(1)=y$
- If X is path connected then X is connected

  (but not always the other way around)