Homework 1

Math 147, Fall 2018

1 Review of sets, functions, equivalence relations

1. Let A and B be sets, both of which have at least two distinct members. Prove that there is a subset $W \subset A \times B$ which is not the Cartesian product of a subset of A with a subset of B.

2. Let $A = \{a_1, a_2\}$ and $B = \{b_1, b_2\}$ be two sets each with two elements. Let $f : A \to B$ be the constant function such that $f(a) = b_1$ for each $a \in A$.

   (a) Prove that $f^{-1}(f(\{a_1\})) \neq \{a_1\}$. [So $f^{-1}(f(X))$ and X are not always equal.]

   (b) Prove that $f(f^{-1}(B)) \neq B$. [So $f(f^{-1(X)})$ and X are not always equal.]

   (c) Prove that $f(\{a_1\} \cap \{a_2\}) \neq f(\{a_1\}) \cap f(\{a_2\})$. [So $f(X \cap X')$ and $f(X) \cap f(X')$ are not always equal.]

3. Let A be a set and $E \subset A$. Define the function $\chi_E : A \to \{0, 1\}$ by setting $\chi_E(x) = 1$ if $x \in E$ and $\chi_E(x) = 0$ if $x \notin E$. If E and F are subsets of A show that

   (a) $\chi_{E \cap F} = \chi_E \cdot \chi_F$

   (b) $\chi_{E \cup F} = \chi_E + \chi_F - \chi_{E \cap F}$

4. Show that the following is an equivalence relation on $\mathbb{R}^2$: $(x_1, y_1) \sim (x_2, y_2)$ if and only if both $x_1 - x_2$ and $y_1 - y_2$ are integers.

2 Metric spaces

5. Show that the following gives a distance function on $\mathbb{R}^n$:

$$d_1(x, y) = \sum_{i=1}^{n} |x_i - y_i|$$

   Draw the unit ball in $\mathbb{R}^2$ using this metric (i.e. the set of points of d_1 distance 1 from the origin $(0, 0)$).

6. Let $C^0([a,b])$ be the set of continuous functions $f : [a,b] \to \mathbb{R}$. Show that if we define

$$d^1(f,g) = \int_a^b |f(t) - g(t)| dt$$

then $(C^0([a,b]), d^1)$ is a metric space.

7. Let X be a set. For any $x, y \in X$ define the function d by

$$d(x,x) = 0$$

and

$$d(x,y) = 1$$

if $x \neq y$. Prove that (X, d) is a metric space.

3 Continuity

8. Show that if we consider the metric space $(\mathbb{R}^2, d_1)$ given in problem 5, then the function $F : (\mathbb{R}^2, d_1) \to (\mathbb{R}^2, d_1)$ defined by $F(x_1, x_2) = (3x_2, 2x_1)$ is continuous function at the origin $(0,0)$ using the ε, δ definition.

9. Show that $f : (X, d_X) \to (Y, d_Y)$ is a continuous function between metric spaces (using the ε, δ definition) if and only if for every open set $U \subset Y$, $f^{-1}(U)$ is open.

10. If X and Y are sets and d is the metric defined in problem 7, and $f : (X, d) \to (Y, d)$ is any function, show that f is continuous.