Homework 2

Math 147, Fall 2018

1. In $(\mathbb{R}^2, d)$ where d is the standard Euclidean metric: $d(x,y) = \sqrt{\sum_i (x_i - y_i)^2}$
   (a) Prove that the unit square $[0,1] \times [0,1]$ is closed.
   (b) Prove that the ball $\{(x_1, x_2) \mid x_1^2 + x_2^2 < 1\}$ is open.
   (c) Prove that the subset $A = \{(x_1, x_2) \mid x_1 < 2\}$ is open.
2. Let X be a metric space. Prove that the union $\cup_\alpha U_\alpha$ of arbitrarily many open sets $\{U_\alpha\}$ is open (using the metric definition of open with open balls).
3. Prove that the intersection $\cap_\alpha V_\alpha$ of arbitrarily many closed sets $\{V_\alpha\}$ is closed (you may use either the limit point definition of closed or the definition that closed is the complement of open).
4. Let $U_n = (-1/n, 1/n)$ be the open interval between $-1/n$ and $1/n$. Verify that U_n is an open subset of $\mathbb{R}$ with the usual metric $d(x,y) = |x - y|$. Determine what is the intersection $U = \cap_{n=1}^{\infty} U_n$ and show that U is not an open set.
5. Give an example of an infinite union $V = \cup_i V_i$ of closed sets $\{V_i\}$ such that V is not closed.
6. Let (X, d_1) and (Y, d_2) be metric spaces. Let $f : X \to Y$ be a *continuous* function. Define a distance function d on $X \times Y$ using the maximum:
   $$d((a_X, a_Y), (b_X, b_Y)) = \max\{d_1(a_X, b_X), d_2(a_Y, b_Y)\}.$$
   Define the graph $\Gamma(f) \subset X \times Y$ to be the subset
   $$\Gamma(f) = \{(x, f(x)) \subset X \times Y\}.$$
   Show that $\Gamma(f)$ is a closed subset of $X \times Y$.
7. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by
   $$f(x) = \begin{cases} \frac{1}{x} & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases}$$
   Show that f is not continuous but $\Gamma(f)$ is still a closed subset of $\mathbb{R}^2$.

8. Let S^1 be the unit circle given by

$$S^1 = \{(x_1, x_2) \mid x_1^2 + x_2^2 = 1\}.$$

Show that S^1 is a closed subset of $\mathbb{R}^2$. [Suggestion: $S^1 = \{x \in \mathbb{R}^2 \mid d_{Euc}(x, 0) = 1\}$, show the complement is open utilizing the triangle inequality.]

Bonus: We have a metric on S^1 coming from the metric on $\mathbb{R}^2$. As in problem 6, we can define a metric on the procut $S^1 \times \mathbb{R}$ using the maximum metric:

$$d(((\cos\theta_1, \sin\theta_1), t_1), ((\cos\theta_2, \sin\theta_2), t_2)) = \max\{d((\cos\theta_1, \sin\theta_1), (\cos\theta_2, \sin\theta_2)), d(t_1, t_2)\}.$$

Let the function $F : \mathbb{R}^2 \setminus \{0\} \to S^1 \times \mathbb{R}$ be defined by

$$F(x_1, x_2) = \left(\left(\frac{x_1}{\sqrt{x_1^2 + x_2^2}}, \frac{x_2}{\sqrt{x_1^2 + x_2^2}} \right), \sqrt{x_1^2 + x_2^2} \right).$$

(a) Prove that F is continuous.

(b) Find a continuous inverse function G for F. Being an inverse means that $G \circ F$ is the identity map on $\mathbb{R}^2 \setminus \{0\}$ and $F \circ G$ is the identity map on $S^1 \times \mathbb{R}$. Prove that these compositions are the identity and that G is continuous.