Homework 3

Math 147, Fall 2018

1 Topological spaces and continuity (3.2,3.5)

1. Suppose X is a topological space. Let A be a subset of X. Suppose that for each $x \in A$ there is an open set U containing x such that $U \subset A$. Show that A is open in X (i.e. A is in the topology).

2. Let $X = \{a, b, c, d\}$ be a set with four elements.

   (a) Show that if we designate the following subsets as the open sets in X, this gives a topology:
   $$\emptyset, \{a, b\}, \{c, d\}, \{a, b, c, d\}$$

   (b) Give $Y = \{0, 1\}$ the discrete topology (every subset is an open set). Show that if $f : X \to Y$ is a continuous function then $f(a) = f(b)$ and $f(c) = f(d)$.

3. Let X and Y be topological spaces, and let $f : X \to Y$ be a continuous function (defined by the fact that the preimage of open sets are open). Show that if $C \subset Y$ is a closed subset then $f^{-1}(C)$ is closed.

2 Basis for a topology and continuity (Munkres §12,13)

4. Verify that the open balls $\{B_\varepsilon(p)\}$ varying over all $\varepsilon \in \mathbb{R}_{>0}$ and $p \in \mathbb{R}^2$ form a basis for the Euclidean topology on $\mathbb{R}^2$.

5. Let X be a set. Give X the *discrete topology* where the collection of open sets $\mathcal{T}_d$ is every subset of X.

   (a) Show that $\mathcal{B}_d = \{\{x\} \mid x \in X\}$, the set of subsets containing a single element of X, forms a basis for the discrete topology.

   (b) Show that every subset $A \subset X$ is both open and closed in the discrete topology.

   (c) Let Y be another topological space and let $f : X \to Y$ be a function. Show that f is continuous.

6. Consider a topology $\mathcal{T}_\ell$ on the real line $\mathbb{R}$ generated by the basis $\mathcal{B}_\ell = \{[a, b) \mid a < b\}$.

   (a) (Update!) Show there exists a closed set C (with respect to this topology) which is bounded from below (i.e. there exists N such that $x \geq N$ for all $x \in C$).

   (b) Let $\mathcal{T}$ denote the usual topology on $\mathbb{R}$ with basis given by open intervals $\{(a, b) \mid a < b\}$, and consider a function

   $$f : (\mathbb{R}, \mathcal{T}_\ell) \to (\mathbb{R}, \mathcal{T}).$$

   Show that if f has right limits equal to the value of the function:

   $$\lim_{x \to a^+} f(x) = f(a)$$

   then f is continuous as a function from $(\mathbb{R}, \mathcal{T}_\ell)$ to $(\mathbb{R}, \mathcal{T})$.

   (c) Consider a different basis where the endpoints of the interval must be rational numbers:

   $$\mathcal{B}_\ell^{\mathbb{Q}} = \{[a, b) \mid a < b \text{ and } a, b \in \mathbb{Q}\}.$$

   Show that the topology $\mathcal{T}_\ell^{\mathbb{Q}}$ generated by this basis is different from the topology $\mathcal{T}_\ell$ by showing that there is at least one open set in one of the topologies that is not an open set in the other topology.

3 Interior, closure, boundary (3.4)

7. In $\mathbb{R}^n$ with the Euclidean topology, let

   $$B = \{(x_1, \cdots, x_n) \mid x_1^2 + \cdots + x_n^2 \leq 1\}$$

   Prove that $(x_1, \cdots, x_n) \in Bdry(B)$ if and only if $x_1^2 + \cdots + x_n^2 = 1$, i.e. the boundary of B is the $(n-1)$ dimensional sphere

   $$S^{n-1} = \{(x_1, \cdots, x_n) \mid x_1^2 + \cdots + x_n^2\}$$

8. In $\mathbb{R}^{n+1}$ with the Euclidean topology, let

   $$P = \{(x_1, \cdots, x_n, x_{n+1}) \mid x_{n+1} = 0\}.$$

   Prove that $Int(P) = \emptyset$, $Bdry(P) = P$, and $\overline{P} = P$.