Homework 4

Math 147, Fall 2018

1 Basis for a topology

1. Let $\mathbb{Z}$ denote the integers and let $\mathcal{B}$ be the collection of subsets of $\mathbb{Z}$ of the form $U = \{an + b \mid n \in \mathbb{Z}\}_{a,b\in\mathbb{Z},\ a\geq 2,\ 0\leq b<a}$.

   (a) Show that $\mathcal{B}$ is a basis for a topology on $\mathbb{Z}$. Let $\tau_{\mathcal{B}}$ denote the topology generated by $\mathcal{B}$.

   (b) Show that each basis element $U_{a,b} = \{an + b \mid n \in \mathbb{Z}\}$ (where $a, b \in \mathbb{Z}$ satisfy $a \geq 2$, $0 \leq b < a$) is both an open set and a closed set in the topology $\tau_{\mathcal{B}}$.

   (c) Consider the basis elements $U_{p,0} = \{pn \mid n \in \mathbb{Z}\}$ where p is a prime number, and let
   $$A = \bigcup_{p \text{ prime}} U_{p,0}.$$
   Show that $\mathbb{Z} \setminus A = \{-1, 1\}$ and use this to prove that A is not closed. Conclude that A is not a finite union of the closed sets $U_{p,0}$, so there must be infinitely many prime numbers.

2 Closure and boundary (3.4)

2. Let $A \subset X$ be a subset of a topological space. Show that $Bdry(A) = \emptyset$ if and only if A is both open and closed in X.

3. A subset $A \subset X$ is said to be *dense* in X if $\overline{A} = X$. Prove that if for every open set $U \subset X$, we have $U \cap A \neq \emptyset$, then A is dense in X.

3 Homeomorphisms (3.5)

4. Suppose $f : X \to Y$ is a homeomorphism of topological spaces. Suppose $h : Y \to Z$ is a function between topological spaces. Show that h is continuous if and only if the composition $h \circ f : X \to Z$ is continuous.

5. Determine which of the following functions are homeomorphisms (using the standard topology on $\mathbb{R}$). If it is a homeomorphism, prove it. If it is not a homeomorphism, explain why.

   (a) $f_1 : \mathbb{R} \to \mathbb{R}$, $f_1(x) = 2x + 3$

   (b) $f_2 : \mathbb{R} \to \mathbb{R}$, $f_2(x) = x^2$

   (c) $f_3 : (0, \infty) \to \mathbb{R}$, $f_3(x) = \ln(x)$

   (d) $f_4 : (0, \infty) \to \mathbb{R}$, $f_4(x) = \sqrt{x}$

6. Prove that an open interval (a, b) considered as a subspace of the real line $\mathbb{R}$ is homeomorphic to the real line $\mathbb{R}$.