Homework 6

Math 147, Fall 2018

1. Let X be the set of natural numbers $1, 2, 3, \cdots$. We define a topology τ on X as follows. If $U \subset X$ does not contain 1 or 2, U is an open set in τ. If $V \subset X$ contains 1 or 2, then it is open if and only if it contains all but finitely many points in X. Show that X with this topology τ is not Hausdorff.

2. Prove that if (X, τ) is a regular topological space and $A \subset X$ and τ_A is the subset topology, show that (A, τ_A) is regular.

3. Let X be the set of natural numbers $1, 2, 3, \cdots$. We define a topology τ' on X as follows. $U \subset X$ is open in τ' if and only if either one or both of the following two conditions hold

   (a) $X \setminus U$ is finite

   (b) $1 \in X \setminus U$

   Prove that X with the topology τ' is normal.

4. Prove that the product of two Hausdorff spaces is Hausdorff.

5. Let $f, g : X \to Y$ be continuous functions between topological spaces, and suppose Y is a Hausdorff space. Show that the subset

$$S = \{x \in X \mid f(x) = g(x)\}$$

   is a closed subset of X.

6. Determine which familiar space is homeomorphic to the following quotient spaces of $\mathbb{R}^2$:

   (a) Under the equivalence relation where $(x_1, y_1) \sim (x_2, y_2)$ if and only if $x_1 = x_2$.

   (b) Under the equivalence relation where $(x_1, y_1) \sim (x_2, y_2)$ if and only if $x_1 + y_1^2 = x_2 + y_2^2$.

   (c) Under the equivalence relation where $(x_1, y_1) \sim (x_2, y_2)$ if and only if $x_1^2 + y_1^2 = x_2^2 + y_2^2$.

7. Define a relation in the plane $\mathbb{R}^2$ by $(x, y) \sim (x', y')$ if and only if $x - x'$ and $y - y'$ are both integers. Prove that $\sim$ is an equivalence relation. Let T be the quotient space of equivalence sets with the quotient (identification) topology and $\phi : \mathbb{R}^2 \to T$ the quotient map.

Bonus: Show that there is a homeomorphism from T to the torus Q in $\mathbb{R}^3$ which is parameterized as follows:

$$Q = \{((a + b\cos\phi)\cos\theta, (a + b\cos\phi)\sin\theta, b\sin\phi) \in \mathbb{R}^3\}$$

8. Let S^1 denote the unit circle:

$$S^1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}.$$

Let D^2 denote the unit disk:

$$D^2 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}.$$

Let S^2 denote the unit sphere:

$$S^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}.$$

(a) Let A be the subset of $S^1 \times [0, 1]$ given by $S^1 \times \{1\}$, i.e. $A = \{(\theta, t) \in S^1 \times [0, 1] \mid t = 1\}$. Show that the quotient $(S^1 \times [0, 1])/A$ is homeomorphic to the disk D^2, by defining a map between them and showing it is continuous and has a continuous inverse.

(b) Let B be the subset of $S^1 \times [0, 1]$ given by $S^1 \times \{0\}$, i.e. $B = \{(\theta, t) \in S^1 \times [0, 1] \mid t = 0\}$. Show that the quotient $(S^1 \times [0, 1])/\sim$ by the equivalence relation $a \sim a'$ if $a, a' \in A$ and $b \sim b'$ if $b, b' \in B$ is homeomorphic to the sphere S^2, by defining a map between them and showing it is continuous and has a continuous inverse.