Homework 8

Math 147, Fall 2018

1. Prove that if a space (X, τ) is path connected then it is connected.

2. A subset $A \subset \mathbb{R}^2$ is called *convex* if for any pair of points $(x_1, y_1), (x_2, y_2) \in A$, the straight line between those two points is completely contained in A.

   (a) Show that if A is a convex subset of $\mathbb{R}^2$ then A is path connected and therefore connected (by the previous problem).

   (b) Show that the unit square $[0, 1] \times [0, 1]$ is path connected and therefore connected.

   (c) Show that the unit disk $\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1\}$ is path connected and therefore connected. [Hint: You may use the triangle inequality for norms:

   $$||av + bw|| \leq a||v|| + b||w||$$

   where v and w are points in $\mathbb{R}^2$, $a, b \in \mathbb{R}$ and the norm $|| \cdot ||$ is defined such that if $v = (x, y)$, $||v|| = \sqrt{x^2 + y^2}$.]

3. Show that $\mathbb{R}$ with the Zariski topology τ_{Zar} is a compact space.

4. Let S be a set and τ_{dis} be the discrete topology on S.

   (a) Show that if S is finite then (S, τ_{dis}) is compact.

   (b) Show that if S is infinite then (S, τ_{dis}) is not compact.

5. Let (X, d) be a compact metric space. Show that X is "bounded with respect to d" meaning show that there exists a number $K > 0$ such that $d(x, y) \leq K$ for all $x, y \in X$.

6. Let $f : (X, \tau) \to (\mathbb{R}, \tau_{Euc})$ be a continuous function, and (X, τ) a compact space. Prove that there is a point $x_0 \in X$ such that $f(x) \leq f(x_0)$ for every $x \in X$, i.e. f has a maximum value on X.

7. Let X be compact, Y be Hausdorff, and $f : X \to Y$ a continuous surjective function. Show that a subset $V \subset Y$ is open in Y if and only if $f^{-1}(V)$ is open in X.

8. In a metric space (X, d) a sequence $a_1, a_2, a_3, \cdots$ of points in X is called a *Cauchy sequence* if for each $\varepsilon > 0$ there is a positive integer N such that $d(a_n, a_m) < \varepsilon$ whenever $n, m > N$. A metric space is called *complete* if every Cauchy sequence in X converges to a point of X. Prove that a compact metric space is complete.