Midterm 1 Solutions

Math 147, Fall 2018

Problem 1: Let X be a set.

(a) For any $x, y \in X$ define the function $d : X \times X \to \mathbb{R}$ by

$$\begin{cases} d(x,x) = 0 & \\ d(x,y) = 1 & \text{if } x \neq y \end{cases}$$

Show that d is a metric.

(1) Positive definite: $d(x,y) \in \{0,1\}$ so $d(x,y) \geq 0$ for all $x, y \in X$. $d(x,y) = 0$ only when $x = y$ by definition.

(2) Symmetric: If $x = y$, $d(x,y) = 0$ and $d(y,x) = 0$. If $x \neq y$ then $d(x,y) = 1$ and $d(y,x) = 1$ so in all cases $d(x,y) = d(y,x)$.

(3) Triangle inequality. If $x, y, z \in X$, we want to show that

$$d(x,z) \leq d(x,y) + d(y,z).$$

To show this, first consider the case when $x = z$. Then $d(x,z) = 0$ and $d(x,y) \geq 0$ and $d(y,z) \geq 0$ so $d(x,z) = 0 \leq d(x,y) + d(y,z)$. Next consider the case when $x \neq z$ so $d(x,z) = 1$.

If $x = y$ then $y \neq z$ so $d(x,y) = 0$ and $d(y,z) = 1$ so $d(x,z) = 1 \leq 0 + 1 = d(x,y) + d(y,z)$.

Similarly, if $y = z$ then $x \neq y$ so $d(x,y) = 1$ and $d(y,z) = 0$ so $d(x,z) = 1 \leq 1 + 0 = d(x,y) + d(y,z)$.

Finally if $x \neq y$ and $y \neq z$ then $d(x,y) = d(y,z) = 1$ so $d(x,z) = 1 \leq 1 + 1 = d(x,y) + d(y,z)$.

(b) Show that every subset $A \subset X$ is an open subset in the metric space topology, so this metric induces the discrete topology on X.

Suppose $A \subset X$. Let $a \in A$ and let $r = 1/2$ so in particular, $r > 0$. Then $B_r(a) = \{x \in X \mid d(x,a) < r = \frac{1}{2}\}$. Since $d(x,a) = 1 > \frac{1}{2}$ whenever $x \neq a$, $B_r(a) = \{a\}$. Therefore $a \in B_r(a) = \{a\} \subset A$ so A is open.

Problem 2: Suppose X is a topological space. Let A be a subset of X. Suppose that for each $x \in A$ there is an open set U_x containing x such that $U \subset A$. Show that A is open in X (i.e. A is in the topology) without quoting any topology theorems we have proven, (you should use the axioms in the definition of a topology and you can use set theory statements).

Let $U = \cup_{x \in A} U_x$. Then U is an open subset of X because each U_x is open in X and U is a union of open subsets.

We first show that $U \subseteq A$. This is true because for each $u \in U$, $u \in U_x$ for some $x \in A$ and $U_x \subset A$. Therefore, $u \in U_x \subset A$ so $u \in A$.

Next we show that $A \subseteq U$. This is true because for each $a \in A$, $a \in U_a$ so $a \in \cup_{x \in A} U_x$.

Therefore $U \subseteq A \subseteq U$ so $A = U$. Since U is open, A is open.

Problem 3: Let τ_{Zar} be the Zariski topology on $\mathbb{R}$, a subset is open $U \in \tau_{Zar}$ if and only if $\mathbb{R} \setminus U$ is a finite set or all of $\mathbb{R}$.

(a) Prove that τ_{Zar} is a topology (satisfies the 3 axioms).

(1) $\emptyset, \mathbb{R} \in \tau_{Zar}$ because $\mathbb{R} \setminus \emptyset = \mathbb{R}$ which is allowed by the definition and $\mathbb{R} \setminus \mathbb{R} = \emptyset$ which has 0 elements so it is a finite set.

(2) If $U_\alpha \in \tau_{Zar}$ for all $\alpha \in \mathcal{I}$ then $\mathbb{R} \setminus U_\alpha$ is a finite set for each α. We want to show that $\cup_{\alpha \in \mathcal{I}} U\alpha$ is open in τ_{Zar}. To show this we use DeMorgan's laws to show that

$$\mathbb{R} \setminus (\bigcup_{\alpha \in \mathcal{I}} U_\alpha) = \bigcap_{\alpha \in \mathcal{I}} (\mathbb{R} \setminus U_\alpha)$$

The intersection of finite sets is finite, therefore $\mathbb{R} \setminus (\cup_\alpha U_\alpha)$ is a finite set so $\cup_\alpha U_\alpha \in \tau_{Zar}$.

(3) If $U_1, \cdots, U_n \in \tau_{Zar}$ then $\mathbb{R} \setminus U_i$ is a finite set for $i = 1, \cdots, n$. Again using DeMorgan's laws we have

$$\mathbb{R} \setminus (U_1 \cap \cdots \cap U_n) = (\mathbb{R} \setminus U_1) \cup \cdots \cup (\mathbb{R} \setminus U_n)$$

The finite union of finite sets is finite so $\mathbb{R} \setminus (U_1 \cap \cdots \cap U_n)$ is finite so $U_1 \cap \cdots \cap U_n \in \tau_{Zar}$.

(b) What is the closure of the subset $(0, 1)$ in the topology τ_{Zar}? Prove it.

The closure of $(0, 1)$ in τ_{Zar} is $\mathbb{R}$. Here is the proof.

The closure of a subset $A \subseteq X$ is the intersection of all closed subsets $C \subseteq X$ such that $A \subseteq C$. A subset $C \subseteq \mathbb{R}$ is closed in the topology τ_{Zar} if and only if $\mathbb{R} \setminus C \in \tau_{Zar}$. $\mathbb{R} \setminus C \in \tau_{Zar}$ if and only if $\mathbb{R} \setminus (\mathbb{R} \setminus C)$ is a finite set or all of $\mathbb{R}$. $C = \mathbb{R} \setminus (\mathbb{R} \setminus C)$ so C is a closed set in τ_{Zar} if and only if C is a finite subset or $C = \mathbb{R}$. Since $(0, 1)$ contains infinitely many elements (for example it contains $1/n$ for all natural numbers n), if $(0, 1) \subset C$, C cannot be a finite set. Therefore the only closed subset of $\mathbb{R}$ in the topology τ_{Zar} such that $(0, 1) \subseteq C$ is when $C = \mathbb{R}$. Therefore $\overline{(0, 1)} = \mathbb{R}$.

Problem 4: Let $X = \{1, 2, 3\}$ with the topology $\tau_X = \{\emptyset, \{1\}, \{2, 3\}, \{1, 2, 3\}\}$. Let $Y = \{A, B, C\}$.

Let $f : X \to Y$ be the bijective function defined by $f(1) = A$, $f(2) = B$ and $f(3) = C$.

(a) Suppose τ_Y is some topology on Y such that the function $f : (X, \tau_X) \to (Y, \tau_Y)$ defined above is continuous using these topologies. Show that $\{B\} \notin \tau_Y$.

Suppose for sake of contradiction that $\{B\} \in \tau_Y$. Then since f is continuous, $f^{-1}(\{B\}) \in \tau_X$. However, looking at the definition of f, we see that the only point which f maps to B is 2. Therefore $f^{-1}(\{B\}) = \{2\}$. Since $\{2\} \notin \tau_X$, we reach a contradiction.

(b) Give an example of a topology τ_Y on Y such that $f : (X, \tau_X) \to (Y, \tau_Y)$ is a continuous function, but is **not** a homeomorphism.

Let $\tau_Y = \{\emptyset, Y\}$ be the trivial topology. Then since $f^{-1}(\emptyset) = \emptyset$ and $f^{-1}(Y) = X$, and $\emptyset, X \in \tau_X$, f is continuous.

f is a bijection so it has an inverse $f^{-1} : Y \to X$ defined by $f^{-1}(A) = 1$, $f^{-1}(B) = 2$ and $f^{-1}(C) = 3$. We will show f^{-1} is not continuous. Consider the open subset $\{1\} \in \tau_X$. Then looking at the definition of the function we see $(f^{-1})^{-1}(\{1\}) = \{A\}$. But $\{A\} \neq Y$ and $\{A\} \neq \emptyset$ so $\{A\} \notin \tau_Y$. Therefore f^{-1} is not continuous so f is not a homeomorphism.